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On the randomization theory of intra-block and inter-block analysis

Tadeusz Caliński and Sanpei Kageyama

Department of Mathematical and Statistical Methods, Academy of Agriculture,
Wojska Polskiego 28, 60-637 Poznań, Poland
Department of Mathematics, Faculty of School Education, Hiroshima University,
Shinonome, Hiroshima 734, Japan

Summary

A general randomization model for experiments in block designs is recalled and conditions are given for obtaining the best linear unbiased estimators under the model. Since the conditions appear to be severely restrictive, a resolution of the model into three effective submodels is considered. Conditions for obtaining the best linear unbiased estimators under these submodels are found. In particular, it is shown under which conditions the best linear unbiased estimators for a contrast of treatment parameters are obtainable from both the intra- and the inter-block analysis, and when they estimate unbiasedly the same contrast. Finally, the efficiency factors of the design for the estimation of a contrast in the two analyses are considered.

1. Introduction and preliminaries

In a series of papers by Kala (1989, 1990, 1991), published in this journal, elements of the randomization theory have been exposed, with special reference to designed experiments. In particular, in the last paper (Kala, 1991) a randomization model for block experiments has been considered and the existence under this model of the best linear unbiased estimators of treatment parametric functions has been discussed. The discussion has also been extended to the submodels that underlie the intra-block and inter-block analysis of Yates (1940). It is the aim of the present paper to continue the discussion and to clarify some points related to these two analyses, the inter-block analysis in particular. Some of the difficulties concerning the inter-block submodel have been considered and explained by Kala (1991, Section 6), but it seems that further results related to this model may be of interest for those who want to utilize the inter-block analysis.

Key words: best linear unbiased estimation, block designs, efficiency factor, inter-block analysis, intra-block analysis, minimum norm quadratic estimation, randomization model

To make the approach adopted in the present paper more comprehensible, some of the basic results given in an earlier paper (Caliński and Kageyama, 1988) have to be repeated. There is, however, a slight difference in defining the intra- and inter-block submodels between the earlier approach (adopted also by Kala, 1991) and that used here. In the present paper the randomization model is resolved not in two, as earlier, but into three submodels. This resolution reflects better the natural stratification of the experimental units, and also is more in the spirit of the general theory of the analysis of randomized experiments introduced by Nelder (1965a, 1965b). This way of resolving a randomization model has already been used in some papers published in this journal, e.g. by Ceranka, Chudzik and Mejza (1991).

The important problem of combining results obtainable from the intra-block and the inter-block analysis is not discussed in the present paper, as it deserves a separate examination and exposition.

The notation and terminology of the present paper follows essentially that used by Pearce (1983, Chapter 3). Thus, a block design is described by its $v\times b$ incidence matrix $\mathbf{N}$, from which $\mathbf{r}=\mathbf{N1}_b$ and $\mathbf{k}=\mathbf{N}'\mathbf{1}_v$ are obtainable as column vectors of treatment replications and of block sizes, respectively, giving $n=\mathbf{1}_v'\mathbf{r}=\mathbf{1}_b'\mathbf{k}$ as the number of units, or plots, used in the experiment ($\mathbf{1}_b$ and $\mathbf{1}_v$ being vectors of ones, of indicated dimensions). The matrix $\mathbf{N}$ can be defined as $\mathbf{N}=\mathbf{\Delta D}'$, where $\mathbf{\Delta}'$ is the $n\times v$ design matrix for treatments, and $\mathbf{D}'$ is the $n\times b$ design matrix for blocks. These matrices provide the diagonal matrices $\mathbf{r}^{\delta}=\mathbf{\Delta\Delta}'$ and $\mathbf{k}^{\delta}=\mathbf{DD}'$ that are used in some formulae, as also their inverses $\mathbf{r}^{-\delta}$ and $\mathbf{k}^{-\delta}$ are. If, by proper ordering, $\mathbf{\Delta}'$ can be written as $\mathbf{\Delta}'=\text{diag}[\mathbf{\Delta}_1':\mathbf{\Delta}_2':...:\mathbf{\Delta}_g']$, in accordance with a partition $v=v_1+v_2+\ldots+v_g$, and, simultaneously, $\mathbf{D}'$ as $\mathbf{D}'=\text{diag}[\mathbf{D}_1':\mathbf{D}_2':\ldots:\mathbf{D}_g']$, in accordance with a partition $b=b_1+b_2+\ldots+b_g$, the design is disconnected for any $g>1$, giving $\mathbf{N}$ as $\mathbf{N}=\text{diag}[\mathbf{N}_1:\mathbf{N}_2:\ldots:\mathbf{N}_g]$. In this notation it is assumed that the $v_l\times b_l$ matrix $\mathbf{N}_l$ $(l=1,2,...,g)$ is the incidence matrix of a connected subdesign. For such a case the equality $\boldsymbol{k}^{\delta}=\text{diag}[k_1\mathbf{I}_{b_1}:k_2\mathbf{I}_{b_2}:...:k_g\mathbf{I}_{b_g}]$ means that the block sizes within each connected subdesign are constant. Then $\mathbf{Nk}^{-\delta}=\widetilde{\mathbf{k}}^{-\delta}\mathbf{N}$, where $\widetilde{\mathbf{k}}^{\delta}=\text{diag}[k_1\mathbf{I}_{v_1}:k_2\mathbf{I}_{v_2}:...:k_g\mathbf{I}_{v_g}]$. Throught the paper it will be assumed that $\mathbf{N}$ is formed in such a way that the above notation applies.

Furthermore, distinction is made between the potential number of blocks, N_B, from which a choice can be made, and the number, b, of those actually chosen for the experiment. The usual situation is that $b=N_B$, but in general $b\leq N_B$. Also, it will be convenient to distinct between the potential (available) number of units within a block, denoted by K (with a subscript), and the number of those actually used in the experiment, denoted by k (with a subscript).

2. A randomization model

According to one of the basic principles of experimental design, the units are to be randomized before they enter the experiment. Suppose that the randomization is performed as described by Nelder (1954), by randomly permuting blocks within a total area of them and by randomly permuting units within the blocks. Then, assuming the usual unit-treatment additivity , and also assuming, as usual, that the technical errors are uncorrelated, with zero expectation and a constant variance, independent of the treatments in particular, the model of the variables observed on the n units actually used in the experiment can be written in matrix notation as

$$\mathbf{y} = \boldsymbol{\Delta}'\boldsymbol{\tau} + \mathbf{D}'\boldsymbol{\beta} + \boldsymbol{\eta} + \mathbf{e}\,, \tag{2.1}$$

where $\mathbf{y}$ is a vector of observed variables, $\boldsymbol{\tau}$ is a vector of treatment parameters, $\boldsymbol{\beta}$ is a vector of block random effects, $\boldsymbol{\eta}$ is a vector of unit errors and $\mathbf{e}$ is a vector of technical errors. Properties of the model (2.1) can be obtained by following its derivation from the randomizations involved.

2.1. Derivation of the model

Suppose that there are N_B blocks, originally labelled $\xi = 1,2,...,N_B$, and that block ξ contains K_ξ units (plots), which are originally labelled $\pi = 1,2,...,K_\xi$. The label may also be written as $\pi(\xi)$, if it is desirable to refer it to block ξ . The randomization of blocks can then be understood as choosing at random a permutation of numbers $1,2,...,N_B$, and then renumbering the blocks with $j = 1,2,...,N_B$ according to the positions of their original labels taken in the random permutation. Similarly, the randomization of units within block ξ can be seen as selecting at random a permutation of numbers $1,2,...,K_\xi$, and then renumbering the units of the block with $l = 1,2,...,K_\xi$ according to the positions of their original labels taken in the random permutation. It will be assumed here that any permutation of block labels can be selected with equal probability, as well as that any permutation of unit labels within a block can be selected with equal probability. Furthermore, it will be assumed that the randomizations of units within the blocks are among the blocks independent, and that they are also independent of the randomization of blocks.

Following Nelder (1965a), it is useful first to consider a model appropriate for analysing experimental data under the assumption that all the $\sum_{\xi=1}^{N_B} K_\xi$ units receive the same treatment, no matter which. The concept of such a "null" experiment makes the derivation of the final model more simple. It needs, however, the assumption that the treatments under consideration are additive in the sense that the variation of the responses among the available experimental units does not depend on the treatment received (see Nelder, 1965b, p. 168; White, 1975, p. 560; also Kala, 1990, p. 36).

For this null experiment let the response of the unit labelled $\pi(\xi)$ be denoted by $\mu_{\pi(\xi)}$, and let it be denoted by $m_{l(j)}$ if by the randomizations the block originally labelled ξ receives label j and the unit originally labelled π in this block receives label l. Introducing the linear identity

$$\mu_{\pi(\xi)} = \mu_{\cdot(\cdot)} + (\mu_{\cdot(\xi)} - \mu_{\cdot(\cdot)}) + (\mu_{\pi(\xi)} - (\mu_{\cdot(\xi)})\,,$$

where (according to the usual dot notation)

$$\mu_{.(\xi)} = K_\xi^{-1} \sum_{\pi(\xi)=1}^{K_\xi} \mu_{\pi(\xi)} \text{ and } \mu_{.(\cdot)} = N_B^{-1} \sum_{\xi=1}^{N_B} \mu_{.(\xi)} \ ,$$

and the variance components defined as (see Nelder, 1977)

$$\sigma_B^2 = (N_B - 1)^{-1} \sum_{\xi=1}^{N_B} (\mu_{.(\xi)} - \mu_{.(\cdot)})^2$$

and

$$\sigma_U^2 = N_B^{-1} \sum_{\xi=1}^{N_B} \sigma_{U,\xi}^2 \ ,$$

with

$$\sigma_{U,\xi}^2 = (K_\xi - 1)^{-1} \sum_{\pi(\xi)=1}^{K_\xi} (\mu_{\pi(\xi)} - \mu_{.(\xi)})^2 \ ,$$

and also the weighted harmonic average K_H defined as

$$K_H^{-1} = N_B^{-1} \sum_{\xi=1}^{N_B} K_\xi^{-1} \sigma_{U,\xi}^2 \,/\, \sigma_U^2 \ ,$$

one can write the linear model

$$m_{l(j)} = \mu + \beta_j + \eta_{l(j)} \ , \qquad (2.2)$$

for any l and j, where $\mu = \mu_{.(\cdot)}$ is a constant parameter, while β_j and $\eta_{l(j)}$ are random variables, the first representing a block random effect, the second a unit error. The following moments of these random variables are easily obtainable:

$$E(\beta_j) = 0 \ , \quad E(\eta_{l(j)}) = 0 \ , \quad \mathrm{Cov}(\beta_j, \eta_{l(j')}) = 0 \quad \text{whether } j=j' \text{ or } j \neq j' \ ,$$

$$\mathrm{Cov}(\beta_j, \beta_{j'}) = \begin{cases} N_B^{-1}(N_B - 1)\sigma_B^2 & \text{if } j=j' \ , \\ -N_B^{-1}\sigma_B^2 & \text{if } j \neq j' \ , \end{cases}$$

and

$$\mathrm{Cov}(\eta_{l(j)}, \eta_{l'(j')}) = \begin{cases} K_H^{-1}(K_H - 1)\sigma_U^2 & \text{if } j=j' \text{ and } l=l' \ , \\ -K_H^{-1}\sigma_U^2 & \text{if } j=j' \text{ and } l \neq l' \ , \\ 0 & \text{if } j \neq j' \ . \end{cases}$$

(For detailed derivations see Caliński and Kageyama, 1988.)

Thus, the responses $\{m_{l(j)}\}$ in the null experiment have the model (2.2) with

$$E(m_{l(j)}) = \mu \tag{2.3}$$

and

$$\text{Cov}(m_{l(j)}, m_{l'(j')}) = (\delta_{jj'} - N_B^{-1})\,\sigma_B^2 + \delta_{jj'}(\delta_{ll'} - K_H^{-1})\sigma_U^2 \ , \tag{2.4}$$

where the δ's are the usual Kronecker deltas, taking value 1 when the indices in the subscript coincide and 0 otherwise.

To continue the derivation, it should be noticed that when observing the responses of the units in reality, any observation may be affected by a "technical error" (see Neyman, Iwaszkiewicz and Kołodziejczyk, 1935; Kempthorne, 1952, pp.132 and 151; Scheffé, 1959, p.293; Ogawa, 1963). Denoting the technical error affecting the observation of the response on the (randomized) unit $l(j)$ by $e_{l(j)}$, the model of the variable observed on that unit in the null experiment can be written as

$$y_{l(j)} = m_{l(j)} + e_{l(j)} = \mu + \beta_j + \eta_{l(j)} + e_{l(j)} \quad , \tag{2.5}$$

for any j and l. It may usually be assumed that the technical errors $\{e_{l(j)}\}$ are uncorrelated, with zero expectation and a constant variance, and that they are independent of the the block effects $\{\beta_j\}$ and of the unit errors $\{\eta_{l(j)}\}$. On account of these assumptions and of the properties established for (2.2), the first and second moments of the random variables $\{y_{l(j)}\}$ defined in (2.5) have the forms

$$E(y_{l(j)}) = \mu$$

and

$$\text{Cov}(y_{l(j)}, y_{l'(j')}) = (\delta_{jj'} - N_B^{-1})\sigma_B^2 + \delta_{jj'}(\delta_{ll'} - K_H^{-1})\sigma_U^2 + \delta_{jj'}\delta_{ll'}\sigma_e^2 \ , \tag{2.6}$$

for all $l(j)$ and all $l'(j')$.

Note that the moments of $\{y_{l(j)}\}$ in the null experiment do not depend on the labels received by the blocks and their units in result of the randomizations. This means that the N_B randomized blocks can be regarded as "homogeneous" and that the set of units randomized within a block can be regarded as such, in the sense that the observed responses of the units may, under the same treatment, be considered as observations on random variables $\{y_{l(j)}\}$ exchangeable within a block and also jointly in sets among the blocks, provided the sets are of a size not exceeding the smallest K_ξ . In fact, to make this concept more feasible one would prefer to have all the K_ξ equal (as demanded by Bailey, 1981, Section 2.2), but this is not necessary for the derivation of the model.

Because of the homogeneity of blocks and that of units within blocks, in the sense given above, the randomization principle can be applied to a block experiment designed according to a chosen incidence matrix **N** by adopting the following rule. The b columns of **N** are assigned to b out of the N_B available blocks of experimental units by assigning the j-th column of **N** to that block which due to the randomization is labelled j. Then the treatments indicated by the j-th column of **N** are assigned to the experimental units of the block labelled j, in numbers defined by the corresponding elements of **N** and in the order determined by the labels the units of the block have received due to the randomization. This rule (which covers that used by Rao, 1959, p. 328) implies not only that $b \le N_B$ but also that the units in the available blocks are in sufficient numbers with regard to the vector $\mathbf{k} = \mathbf{N}'\mathbf{1}_v$ (see also Kala, 1991,p. 14). This means that either the choice of **N** is to be conditioned by the constraint that none of its k_j's exceeds the smallest K_ξ , or an adjustment of **N** is to be made after the randomization of blocks (as suggested by White, 1975, p.558).

Now, adopting the assumption of complete additivity, as mentioned earlier, i.e. assuming that the variances and covariances of the random variables $\{\beta_j\}, \{\eta_{l(j)}\}$ and $\{e_{l(j)}\}$ do not depend on the treatment applied, the adjustment of the model (2.6) to a real situation of comparing several treatments in the same experiment can be made by changing the constant term only. Thus, the model gets the form

$$y_{l(j)}(i) = \mu(i) + \beta_j + \eta_{l(j)} + e_{l(j)} \tag{2.7}$$

$$(i = 1,2,...,v; \;\; j = 1,2,...,b; \;\; l(j) = 1,2,...,k_j),$$

with

$$\mathrm{E}[y_{l(j)}(i)] = \mu(i) = N_B^{-1} \sum_{\xi=1}^{N_B} K_\xi^{-1} \sum_{\pi(\xi)=1}^{K_\xi} \mu_{\pi(\xi)}(i), \tag{2.8}$$

where $\mu_{\pi(\xi)}(i)$ is the true response of unit π in block ξ to treatment i, and with

$$\mathrm{Cov}[y_{l(j)}(i),\; y_{l'(j')}(i')] = \mathrm{Cov}(y_{l(j)}\,,\; y_{l'(j')}), \tag{2.9}$$

as given in (2.6).

Finally, writing the observed variables $\{y_{l(j)}(i)\}$ in form of an $n\times 1$ vector **y**, and the corresponding unit error and technical error variables in form of $n\times 1$ vectors $\boldsymbol{\eta}$ and **e**, respectively, and also writing the treatment parameters as $\boldsymbol{\tau} = [\tau_1, \tau_2, ..., \tau_v]'$, where $\tau_i = \mu(i)$, and the block variables as $\boldsymbol{\beta} = [\beta_1, \beta_2, ..., \beta_b]'$, one can express the model (2.7) in the matrix notation as given in (2.1), and the corresponding moments (2.8) and (2.9) in form of the expectation vector

$$\mathrm{E}(\mathbf{y}) = \boldsymbol{\Delta}'\boldsymbol{\tau} \tag{2.10}$$

and the dispersion matrix (covariance matrix)

$$\mathrm{Cov}(\mathbf{y}) = (\mathbf{D}'\mathbf{D} - N_B^{-1}\mathbf{1}_n\mathbf{1}_n')\sigma_B^2 + (\mathbf{I_n} - K_H^{-1}\mathbf{D}'\mathbf{D})\sigma_U^2 + \mathbf{I}_n\sigma_e^2\,, \tag{2.11}$$

respectively.

The model (2.1), with properties (2.10) and (2.11), coincides with that of Patterson and Thompson (1971) when their matrix $\boldsymbol{\Gamma}$ in the formula

$$\mathrm{Cov}(\mathbf{y}) = (\mathbf{D}'\mathbf{\Gamma}\mathbf{D} + \mathbf{I}_n)\sigma^2 \tag{2.12}$$

is taken equal to $\mathbf{I}_n\gamma - N_B^{-1}\mathbf{1}_b\mathbf{1}_b'\sigma_B^2 / \sigma^2$, where $\gamma = (\sigma_B^2 - K_H^{-1}\sigma_U^2)/\sigma^2$ and $\sigma^2 = \sigma_U^2 + \sigma_e^2$. In fact they have considered a simplified model with $\mathbf{\Gamma} = \mathbf{I}_n\gamma$ in (2.16). Furthermore, if $k_1 = k_2 = \ldots = k_b = k$ (say), $b = N_B$ and $k = K_H$ (the latter implying the equality of all K_ξ), then the present model becomes equivalent to that considered by Rao (1959) and by Roy and Shah (1962), except that the latter authors do not take the technical error into account. Also, in its general form, the present model coincides exactly with that recently obtained by Kala (1991, Section 5) under more general considerations.

2.2. *Main estimation results*

Under the present model the following main results concerning the linear estimation of treatment parametric functions are obtainable.

Theorem 2.1. Under the model (2.1), with its properties (2.10) and (2.11), a function $\mathbf{w}'\mathbf{y}$ is uniformly the best linear unbiased estimator (BLUE) of $\mathbf{c}'\boldsymbol{\tau}$ if and only if $\mathbf{w} = \mathbf{\Delta}'\mathbf{s}$, where $\mathbf{s} = \mathbf{r}^{-\delta}\mathbf{c}$ satisfies the condition

$$(\mathbf{k}^{\delta} - \mathbf{N}'\mathbf{r}^{-\delta}\mathbf{N})\mathbf{N}'\mathbf{s} = \mathbf{0} \tag{2.13}$$

Proof. By Theorem 3 of Zyskind (1967), a function $\mathbf{w}'\mathbf{y}$ is, under the considered model, the BLUE of its expectation, i.e. of $\mathbf{w}'\mathbf{\Delta}'\boldsymbol{\tau}$, if and only if

$$(\mathbf{I}_n - \mathbf{P}_{\Delta'})[(\mathbf{D}'\mathbf{D} - N_B^{-1}\mathbf{1}_n\mathbf{1}_n')\sigma_B^2 + (\mathbf{I}_n - K_H^{-1}\mathbf{D}'\mathbf{D})\sigma_U^2 + \mathbf{I}_n\sigma_e^2]\mathbf{w} = \mathbf{0}, \tag{2.14}$$

where $\mathbf{P}_{\Delta'} = \mathbf{\Delta}'\mathbf{r}^{-\delta}\mathbf{\Delta}$ denotes the orthogonal projector on $\mathcal{C}(\mathbf{\Delta}')$, the column space of $\mathbf{\Delta}'$. If this is to hold uniformly for any values of the variance components σ_B^2, σ_U^2 and σ_e^2, it is necessary and sufficient that $(\mathbf{I}_n - \mathbf{P}_{\Delta'})\mathbf{w} = \mathbf{0}$ and $(\mathbf{I}_n - \mathbf{P}_{\Delta'})\mathbf{D}'\mathbf{D}\mathbf{w} = \mathbf{0}$. But these equations hold simultaneously if and only if $\mathbf{w} = \mathbf{\Delta}'\mathbf{s}$ and $(\mathbf{I}_n - \mathbf{P}_{\Delta'})\mathbf{D}'\mathbf{D}\mathbf{\Delta}'\mathbf{s} = \mathbf{0}$ for some $\mathbf{s}$, the latter equation being equivalent to (2.13). ❑

In connection with this proof it may be noted that the component $N_B^{-1}\mathbf{1}_n\mathbf{1}_n'\sigma_B^2$ in (2.11) does not play any role in establishing Theorem 2.1, since $(\mathbf{I}_n - \mathbf{P}_{\Delta'})\mathbf{1}_n = \mathbf{0}$. For the same reason the simplification of $\mathbf{\Gamma}$ in (2.12) to the form $\mathbf{I}_n\gamma$, made by Patterson and Thompson (1971), does not affect the BLUE of $\mathbf{c}'\boldsymbol{\tau}$. (For more general considerations see Kala, 1981, Theorem 6.2).

Corollary 2.1. For the estimation of $\mathbf{c}'\boldsymbol{\tau} = \mathbf{s}'\mathbf{r}^{\delta}\boldsymbol{\tau}$ under the model as in Theorem 2.1, the following applies.

(a) If $\mathbf{N}'\mathbf{s} = \mathbf{0}$, then (2.13) is satisfied and the estimated function is a contrast, i.e., $\mathbf{c}'\mathbf{1}_v = \mathbf{s}'\mathbf{r} = 0$.

(b) If $\mathbf{N}'\mathbf{s} \neq \mathbf{0}$, then to satisfy (2.13) it is necessary and sufficient that the elements of $\mathbf{N}'\mathbf{s}$ obtained from the same connected subdesign are all equal, i.e., that

$$\mathbf{N's} \in C\,\{\mathrm{diag}[\mathbf{1}_{b_1},\mathbf{1}_{b_2},...,\mathbf{1}_{b_g}]\}\,.$$

Proof. Result (a) is obvious, with $\mathbf{c}'\mathbf{1}_v = 0$ from $\mathbf{N}\mathbf{1}_b = \mathbf{r}$. To prove (b), note that (2.13) is equivalent to the condition $\mathbf{N}'\mathbf{r}^{-\delta}\mathbf{N}\mathbf{N}'\mathbf{s} = \mathbf{k}^{\delta}\mathbf{N}'\mathbf{s}$, which in case of a connected design is satisfied if and only if $\mathbf{N's} \in C\,(\mathbf{1}_b)$. A straightforward generalization of this leads to the result, applicable to any $\mathbf{N} = \mathrm{diag}[\mathbf{N}_1, \mathbf{N}_2,...,\mathbf{N}_g]$ with $g \geq 1$. ❑

Now a question arises, under which design conditions any function $\mathbf{s}'\Delta\mathbf{y}$ is the BLUE of its expectation. This can be answered as follows.

Theorem 2.2. Under the model as in Theorem 2.1, any function $\mathbf{w}'\mathbf{y} = \mathbf{s}'\Delta\mathbf{y}$, i.e. with any s, is uniformly the BLUE of $\mathrm{E}(\mathbf{w}'\mathbf{y}) = \mathbf{s}'\mathbf{r}^{\delta}\boldsymbol{\tau}$ if and only if

(i) the design is orthogonal (in the sense of Darroch and Silvey, 1963) and

(ii) the block sizes of the design are constant within any of its connected subdesigns.

Proof. From the proof of Theorem 2.1 it is evident that $\mathbf{s}'\Delta\mathbf{y}$ is the BLUE of its expectation for any s if and only if

$$(\mathbf{I}_n - \mathbf{P}_{\Delta'})\mathbf{D}'\mathbf{D}\Delta' = \mathbf{0}. \tag{2.15}$$

The equality (2.15), however, implies that $\mathbf{k}^{\delta}\mathbf{N}' = \mathbf{N}'\mathbf{r}^{-\delta}\mathbf{N}\mathbf{N}'$, from which $\mathbf{k}$ is to be an eigenvector of $\mathbf{N}'\mathbf{r}^{-\delta}\mathbf{N}$ with respect to $\mathbf{k}^{\delta}$ corresponding to the eigenvalue 1. But, as it follows from the proof of Corollary 2.1(b), for this it is necessary and sufficient that condition (ii) holds. On the other hand, if (ii) is satisfied, then the matrix $\tilde{\mathbf{k}}^{\delta}$ (defined in Section 1) can be used to show that the equality (2.15) is equivalent to the equality $(\mathbf{I}_n - \mathbf{P}_{\Delta'})\mathbf{D}'\tilde{\mathbf{k}}^{-\delta}\mathbf{D}\Delta' = \mathbf{0}$, which in turn can be shown (see Seber, 1980, Section 6.2) to be equivalent to (i). Thus, (2.15) implies (ii) and (i), subsequently, and, vice versa, (i) and (ii) imply (2.15). ❑

Remark 2.1. Note that condition (i) of Theorem 2.2 is equivalent to the condition $\mathbf{N} = \mathbf{N}\mathbf{k}^{-\delta}\mathbf{N}'\mathbf{r}^{-\delta}\mathbf{N}$ given by Chakrabarti (1962), which for a connected design reduces to $\mathbf{N} = n^{-1}\mathbf{r}\mathbf{k}'$, as given by Pearce (1970). If (ii) of Theorem 2.2 holds, then the orthogonality condition can be written as $\mathbf{N} = \tilde{\mathbf{k}}^{-\delta}\mathbf{N}\mathbf{N}'\mathbf{r}^{-\delta}\mathbf{N}$ in general, and as $\mathbf{N} = (k/n)\mathbf{r}\mathbf{1}_b'$ for a connected design. Also note that the two conditions of Theorem 2.2 coincide with condition (5.12) of Theorem 3 given by Kala (1991).

Remark 2.2. If conditions (i) and (ii) stated in Theorem 2.2 are satisfied, i.e., if (2.15) holds, then

$$\mathrm{Cov}(\mathbf{y})\,\Delta' = \Delta'[(\mathbf{r}^{-\delta}\mathbf{N}\mathbf{N}' - N_B^{-1}\mathbf{1}_v\mathbf{r}')\sigma_B^2 + (\mathbf{I}_v - K_H^{-1}\mathbf{r}^{-\delta}\mathbf{N}\mathbf{N}')\sigma_U^2 + \mathbf{I}_v\sigma_e^2],$$

which implies that both $\Delta'\mathbf{s}$ and $\mathrm{Cov}(\mathbf{y})\Delta'\mathbf{s}$ belong to $C\,(\Delta')$ for any s, and thus, by Theorem 4 of Zyskind (1967), the BLUEs obtainable under the model (2.1), with the moments (2.10) and (2.11), can equivalently be obtained under a simple alternative model in which the covariance matrix (2.11) is replaced by the identity matrix $\mathbf{I}_n$ multiplied by a positive scalar. [See also Rao and Mitra , 1971, Section 8.1]. Moreover, it can be shown (applying, e.g., Theorem 2.3.2 of Rao and Mitra, 1971) that the equality (2.15) is not only a sufficient but also a necessary condition for the BLUEs obtainable under the two alternative models to be the same. In other words, (2.15) is a necessary and sufficient condition for $\mathbf{s}'\mathbf{\Delta y}$ to be both the simple least squares estimator (SLSE) and the BLUE of its expectation, $\mathbf{s}'\mathbf{r}^{\delta}\boldsymbol{\tau}$, whichever vector **s** is used. This coincides virtually with Theorem 3 of Kala (1991).

3. Resolving into effective submodels

Results of Section 2 sound discouraging, as in many designs the BLUEs will exist under the model (2.1) for only a few parametric functions of interest, or for none of them. For example, in the case of a balanced incomplete block (BIB) design none of the contrasts of treatment parameters will have the BLUE (on account of Corollary 2.1).

The apparent difficulty with the model (2.1) can be evaded by resolving it into three submodels (two for the contrasts), in accordance with the stratification of the experimental units. In fact, the units of a block experiment can be seen as being grouped according to a nested classification with three "strata". Adopting the terminology used by Pearce (1983, p.109), these strata may be specified as follows:

1st stratum – of units within blocks, called "intra-block",

2nd stratum – of blocks within the total area, called "inter-block",

3rd stratum – of the total area.

Due to this stratification, the observed vector **y** can be decomposed as

$$\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2 + \mathbf{y}_3 \ , \tag{3.1}$$

where each of the three components is related to one of the strata. The component vectors $\mathbf{y}_\alpha$, $\alpha = 1,2,3$, are thus obtainable by projecting **y** orthogonally on relevant subspaces, mutually orthogonal. The first component in (3.1) can be written as

$$\mathbf{y}_1 = \boldsymbol{\varphi}_1\mathbf{y} \ , \tag{3.2}$$

where

$$\boldsymbol{\varphi}_1 = \mathbf{I}_n - \mathbf{D}'\mathbf{k}^{-\delta}\mathbf{D} = \mathbf{P}_{(\mathbf{D}')^{\perp}} \tag{3.3}$$

(so $\boldsymbol{\varphi}_1 \equiv \boldsymbol{\varphi}$ in the notation of Pearce 1983, Section 3.1), i.e., $\mathbf{y}_1$ is the orthogonal projection of **y** on $\mathcal{C}^{\perp}(\mathbf{D}')$, the orthogonal complement of $\mathcal{C}(\mathbf{D}')$. The second component is

$$\mathbf{y}_2 = \boldsymbol{\varphi}_2\mathbf{y} \ , \tag{3.4}$$

where

$$\boldsymbol{\varphi}_2 = \mathbf{D}'\mathbf{k}^{-\delta}\mathbf{D} - n^{-1}\mathbf{1}_n\mathbf{1}' = \mathbf{P}_{\mathbf{D}'} - \mathbf{P}_{\mathbf{1}_n} \ , \tag{3.5}$$

i.e., $\mathbf{y}_2$ is the orthogonal projection of **y** on $\mathcal{C}^{\perp}(\mathbf{1}_n)\cap\mathcal{C}(\mathbf{D}')$, the orthogonal complement of $\mathcal{C}(\mathbf{1}_n)$ in $\mathcal{C}(\mathbf{D}')$. The third is

$$\mathbf{y}_3 = \boldsymbol{\varphi}_3\mathbf{y}\ , \tag{3.6}$$

where

$$\boldsymbol{\varphi}_3 = n^{-1}\mathbf{1}_n\mathbf{1}_n' = \mathbf{P}_{\mathbf{1}_n}\ , \tag{3.7}$$

i.e., $\mathbf{y}_3$ is the orthogonal projection of $\mathbf{y}$ on $\mathcal{C}(\mathbf{1}_n)$. Evidently, the three matrices (3.3), (3.5) and (3.7) satisfy the conditions

$$\boldsymbol{\varphi}_\alpha = \boldsymbol{\varphi}_\alpha'\ ,\ \boldsymbol{\varphi}_\alpha\boldsymbol{\varphi}_\alpha = \boldsymbol{\varphi}_\alpha\ ,\ \boldsymbol{\varphi}_\alpha\boldsymbol{\varphi}_{\alpha'} = \mathbf{0}\ \text{ for }\ \alpha \neq \alpha'\ , \tag{3.8}$$

where $\alpha, \alpha' = 1,2,3$, and the condition

$$\boldsymbol{\varphi}_1 + \boldsymbol{\varphi}_2 + \boldsymbol{\varphi}_3 = \mathbf{I}_n\ , \tag{3.9}$$

the third equality in (3.8) implying, in particular, that

$$\boldsymbol{\varphi}_1\mathbf{D}' = \mathbf{0}\ \text{ and }\ \boldsymbol{\varphi}_\alpha\mathbf{1}_n = \mathbf{0}\ \text{ for }\ \alpha = 1,2. \tag{3.10}$$

The resulting projections (3.2), (3.4) and (3.6) can be considered as submodels of the overall model (2.1). They are of particular interest when the condition (2.13) is not satisfied. The submodel (3.2) leads to the intra-block analysis, while (3.4) to the inter-block analysis . The submodel (3.6) underlies the total-area analysis, suitable mainly for estimating the general parametric mean.

3.1. The intra-block submodel

The submodel (3.2) has the properties

$$\mathrm{E}(\mathbf{y}_1) = \boldsymbol{\varphi}_1\boldsymbol{\Delta}'\boldsymbol{\tau} = \boldsymbol{\Delta}'\boldsymbol{\tau} - \mathbf{D}'\mathbf{k}^{-\delta}\mathbf{D}\boldsymbol{\Delta}'\boldsymbol{\tau} \tag{3.11}$$

and

$$\mathrm{Cov}(\mathbf{y}_1) = \boldsymbol{\varphi}_1(\sigma_U^2 + \sigma_e^2)\ . \tag{3.12}$$

From them, the main result concerning estimation under (3.2) can be stated as follows.

Theorem 3.1. Under (3.2), a function $\mathbf{w}'\mathbf{y}_1 = \mathbf{w}'\boldsymbol{\varphi}_1\mathbf{y}$ is uniformly the BLUE of $\mathbf{c}'\boldsymbol{\tau}$ if and only if $\boldsymbol{\varphi}_1\mathbf{w} = \boldsymbol{\varphi}_1\boldsymbol{\Delta}'\mathbf{s}$, where the vectors $\mathbf{c}$ and $\mathbf{s}$ are in the relation $\mathbf{c} = \boldsymbol{\Delta}\boldsymbol{\varphi}_1\boldsymbol{\Delta}'\mathbf{s}$ (i.e. $\mathbf{c} = \mathbf{C}\mathbf{s}$ in the notation of Pearce, 1983, Sections 3.1-3.3, where $\boldsymbol{\Delta}\boldsymbol{\varphi}_1\boldsymbol{\Delta}' = \mathbf{r}^\delta - \mathbf{N}\mathbf{k}^{-\delta}\mathbf{N}'$ is denoted by $\mathbf{C}$, as usual).

Proof. Under (3.2), with (3.11) and (3.12), the necessary and sufficient condition for a function $\mathbf{w}'\mathbf{y}_1 = \mathbf{w}'\boldsymbol{\varphi}_1\mathbf{y}$ to be the BLUE of $\mathrm{E}(\mathbf{w}'\mathbf{y}_1) = \mathbf{w}'\boldsymbol{\varphi}_1\boldsymbol{\Delta}'\boldsymbol{\tau}$ is, on account of Theorem 3 of Zyskind (1967), the equality

$$(\mathbf{I}_n - \mathbf{P}_{\boldsymbol{\varphi}_1\boldsymbol{\Delta}'})\boldsymbol{\varphi}_1\mathbf{w} = \mathbf{0}\ \ .$$

It is satisfied if and only if $\boldsymbol{\varphi}_1\mathbf{w} = \boldsymbol{\varphi}_1\boldsymbol{\Delta}'\mathbf{s}$ for some s. But $E(\mathbf{s}'\boldsymbol{\Delta}\mathbf{y}_1) = \mathbf{s}'\boldsymbol{\Delta}\boldsymbol{\varphi}_1\boldsymbol{\Delta}'\boldsymbol{\tau}$. Hence the relation for c and s follows. □

Remark 3.1. Since $\mathbf{1}_v{}'\boldsymbol{\Delta}\boldsymbol{\varphi}_1 = \mathbf{0}$, from (3.10), the only parametric functions for which the BLUEs may exist under (3.2) are contrasts.

If $\mathbf{c}'\boldsymbol{\tau}$ is a contrast, and the condition of Theorem 3.1 is satisfied, then the variance of its BLUE under (3.2), i.e. of $\widehat{\mathbf{c}'\boldsymbol{\tau}} = \mathbf{s}'\boldsymbol{\Delta}\mathbf{y}_1$, is of the form

$$\mathrm{Var}(\widehat{\mathbf{c}'\boldsymbol{\tau}}) = \mathbf{s}'\boldsymbol{\Delta}\boldsymbol{\varphi}_1\boldsymbol{\Delta}'\mathbf{s}(\sigma_U^2 + \sigma_e^2) = \mathbf{c}'(\boldsymbol{\Delta}\boldsymbol{\varphi}_1\boldsymbol{\Delta}')^-\mathbf{c}(\sigma_U^2 + \sigma_e^2) \ , \tag{3.13}$$

where $(\boldsymbol{\Delta}\boldsymbol{\varphi}_1\boldsymbol{\Delta}')^-$ is any generalized inverse (g-inverse) of the matrix $\mathbf{C} = \boldsymbol{\Delta}\boldsymbol{\varphi}_1\boldsymbol{\Delta}'$ (see also Pearce, 1983, p. 62).

Remark 3.2. Since $\mathrm{Cov}(\mathbf{y}_1)\boldsymbol{\varphi}_1\boldsymbol{\Delta}' = \boldsymbol{\varphi}_1\boldsymbol{\Delta}'(\sigma_U^2 + \sigma_e^2)$, which implies that both $\boldsymbol{\varphi}_1\boldsymbol{\Delta}'\mathbf{s}$ and $\mathrm{Cov}(\mathbf{y}_1)\boldsymbol{\varphi}_1\boldsymbol{\Delta}'\mathbf{s}$ belong to $\mathcal{C}(\boldsymbol{\varphi}_1\boldsymbol{\Delta}')$ for any s, it follows from Theorem 3.1, on account of Theorem 4 of Zyskind (1967), that the BLUEs under the submodel (3.2), with the moments (3.11) and (3.12), can equivalently be obtained under a simple alternative model in which the covariance matrix (3.12) is replaced by that of the form $\mathbf{I}_n(\sigma_U^2 + \sigma_e^2)$, i.e., that $\mathbf{s}'\boldsymbol{\Delta}\boldsymbol{\varphi}_1\mathbf{y}$ is both the SLSE and the BLUE of its expectation, for any s. (See also Kala, 1991, p. 20.)

From Remark 3.2 it follows, in particular, that the BLUE of the expectation vector (3.11) can be obtained by a simple least squares procedure (see, e.g., Seber, 1980, Chapter 3), in the form

$$\hat{E}(\mathbf{y}_1) = \mathbf{P}_{\boldsymbol{\varphi}_1\boldsymbol{\Delta}'}\mathbf{y}_1 \ ,$$

where $\mathbf{P}_{\boldsymbol{\varphi}_1\boldsymbol{\Delta}'} = \boldsymbol{\varphi}_1\boldsymbol{\Delta}'(\boldsymbol{\Delta}\boldsymbol{\varphi}_1\boldsymbol{\Delta}')^-\boldsymbol{\Delta}\boldsymbol{\varphi}_1 = \boldsymbol{\varphi}_1\boldsymbol{\Delta}'\mathbf{C}^-\boldsymbol{\Delta}\boldsymbol{\varphi}_1$. Furthermore, it follows that the vector $\mathbf{y}_1$ can be decomposed as

$$\begin{aligned}\mathbf{y}_1 &= \mathbf{P}_{\boldsymbol{\varphi}_1\boldsymbol{\Delta}'}\mathbf{y}_1 + (\mathbf{I}_n - \mathbf{P}_{\boldsymbol{\varphi}_1\boldsymbol{\Delta}'})\mathbf{y}_1 \qquad \text{(in terms of } \mathbf{y}_1\text{)}\\ &= \mathbf{P}_{\boldsymbol{\varphi}_1\boldsymbol{\Delta}'}\mathbf{y} + (\boldsymbol{\varphi}_1 - \mathbf{P}_{\boldsymbol{\varphi}_1\boldsymbol{\Delta}'})\mathbf{y} \qquad \text{(in terms of } \mathbf{y}\text{)}.\end{aligned}$$

(See also Rao, 1974, Section 3.)

Taking the squared norm on both sides of the above decomposition of $\mathbf{y}_1$, one can write

$$\|\mathbf{y}_1\|^2 = \|\mathbf{P}_{\boldsymbol{\varphi}_1\boldsymbol{\Delta}'}\mathbf{y}_1\|^2 + \|(\mathbf{I}_n - \mathbf{P}_{\boldsymbol{\varphi}_1\boldsymbol{\Delta}'})\mathbf{y}_1\|^2.$$

This provides the intra-block analysis of variance, which in terms of the observed vector y can be expressed in a more customary way as

$$\mathbf{y}'\boldsymbol{\varphi}_1\mathbf{y} = \mathbf{y}'\boldsymbol{\varphi}_1\boldsymbol{\Delta}'\mathbf{C}^-\boldsymbol{\Delta}\boldsymbol{\varphi}_1\mathbf{y} + \mathbf{y}'(\boldsymbol{\varphi}_1 - \boldsymbol{\varphi}_1\boldsymbol{\Delta}'\mathbf{C}^-\boldsymbol{\Delta}\boldsymbol{\varphi}_1)\mathbf{y} = \mathbf{Q}_1{}'\mathbf{C}^-\mathbf{Q}_1 + \mathbf{y}'\boldsymbol{\psi}_1\mathbf{y},$$

where $\mathbf{Q}_1 = \boldsymbol{\Delta}\boldsymbol{\varphi}_1\mathbf{y}$ ($\equiv \mathbf{Q}$ in the notation of Pearce, 1983, Section 3.1) and $\boldsymbol{\psi}_1 = \boldsymbol{\varphi}_1\boldsymbol{\Delta}'\mathbf{C}^-\boldsymbol{\Delta}\boldsymbol{\varphi}_1 = \boldsymbol{\varphi}_1(\mathbf{I}_n - \boldsymbol{\Delta}'\mathbf{C}^-\boldsymbol{\Delta})\boldsymbol{\varphi}_1$ ($\equiv\boldsymbol{\psi}$, the "residual matrix" used by Pearce, 1983, Section 3.3). The quadratic form $\mathbf{y}'\boldsymbol{\varphi}_1\mathbf{y}$ can be called the intra-block total sum of squares, and its components, $\mathbf{Q}_1'\mathbf{C}^-\mathbf{Q}_1$ and $\mathbf{y}'\boldsymbol{\psi}_1\mathbf{y}$, can then be called the intra-block treatment sum of squares and the intra-block residual sum of squares, respectively. The corresponding degrees of freedom (d.f.) are $n-b=\text{rank}(\boldsymbol{\varphi}_1)$ for the total, $h=\text{rank}(\mathbf{C})$ for the treatment component and $n-b-h=\text{rank}(\boldsymbol{\psi}_1)$ for the residual component. It can easily be proved (by , e.g., Theorem 9.4.1 of Rao and Mitra, 1971) that the two component sums of squares are distributed independently. The expectations of these component sums of squares are [according to formula (4a. 1.7) in Rao, 1973]

$$\mathrm{E}(\mathbf{Q}_1'\mathbf{C}^-\mathbf{Q}_1)=h(\sigma_U^2+\sigma_e^2)+\boldsymbol{\tau}'\mathbf{C}\boldsymbol{\tau}$$

and

$$\mathrm{E}(\mathbf{y}'\boldsymbol{\psi}_1\mathbf{y})=(n-b-h)(\sigma_U^2+\sigma_e^2).$$

It follows from the latter that the intra-block residual mean square $s_1^2=\mathbf{y}'\boldsymbol{\psi}_1\mathbf{y}/(n-b-h)$ is an unbiased estimator of $\sigma_1^2=\sigma_U^2+\sigma_e^2$. Moreover, s_1^2 is the minimum norm quadratic unbiased estimator (MINQUE) of σ_1^2 under the submodel (3.2), as it may be seen from Theorem 3.4 of Rao (1974).

Thus, s_1^2 can be used to obtain an unbiased estimator of the variance (3.13), in the form

$$\widehat{\mathrm{Var}}(\mathbf{c}'\hat{\boldsymbol{\tau}})=\mathbf{s}'\mathbf{C}\mathbf{s}s_1^2=\mathbf{c}'\mathbf{C}^-\mathbf{c}s_1^2.$$

Furthermore, since under the multivariate normal distribution of $\mathbf{y}$, and hence of $\mathbf{y}_1$, both $\mathbf{Q}_1'\mathbf{C}^-\mathbf{Q}_1/\sigma_1^2$ and $\mathbf{y}'\boldsymbol{\psi}_1\mathbf{y}/\sigma_1^2$ have χ^2 distributions, the first on h d.f. with the noncentrality parameter $\delta=\boldsymbol{\tau}'\mathbf{C}\boldsymbol{\tau}/\sigma_1^2$, the second on $n-b-h$ d.f. with $\delta=0$ (as can be proved applying, e.g., Theorem 9.2.1 of Rao and Mitra, 1971), the hypothesis $\boldsymbol{\tau}'\mathbf{C}\boldsymbol{\tau}=0$, equivalent to $\mathrm{E}(\mathbf{y}_1)=\mathbf{0}$ [or $\mathrm{E}(\mathbf{y})\in\mathcal{C}(\mathbf{D}')$], can be tested by the variance ratio criterion

$$h^{-1}\mathbf{Q}_1'\mathbf{C}^-\mathbf{Q}_1/s_1^2,$$

which under the normality assumption has then the F distribution with h and $n-b-h$ d.f., central when the hypothesis is true.

3.2. The inter-block submodel

As to the submodel (3.4), it has the properties

$$\mathrm{E}(\mathbf{y}_2)=\boldsymbol{\varphi}_2\boldsymbol{\Delta}'\boldsymbol{\tau}=\mathbf{D}'\mathbf{k}^{-\delta}\mathbf{D}\boldsymbol{\Delta}'\boldsymbol{\tau}-n^{-1}\mathbf{1}_n\mathbf{r}'\boldsymbol{\tau} \tag{3.14}$$

and

$$\mathrm{Cov}(\mathbf{y}_2)=(\mathbf{I}_n-n^{-1}\mathbf{1}_n\mathbf{1}_n')\mathbf{D}'\mathbf{D}(\mathbf{I}_n-n^{-1}\mathbf{1}_n\mathbf{1}_n')(\sigma_B^2-K_H^{-1}\sigma_U^2)+\boldsymbol{\varphi}_2(\sigma_U^2+\sigma_e^2)$$

$$= \boldsymbol{\varphi}_2\mathbf{D}'\mathbf{D}\boldsymbol{\varphi}_2(\sigma_B^2 - K_H^{-1}\sigma_U^2) + \boldsymbol{\varphi}_2(\sigma_U^2 + \sigma_e^2). \tag{3.15}$$

The main estimation result under (3.4) can be expressed as follows.

Theorem 3.2. Under (3.4), a function $\mathbf{w}'\mathbf{y}_2 = \mathbf{w}'\boldsymbol{\varphi}_2\mathbf{y}$ is uniformly the BLUE of $\mathbf{c}'\boldsymbol{\tau}$ if and only if $\boldsymbol{\varphi}_2\mathbf{w} = \boldsymbol{\varphi}_2\boldsymbol{\Delta}'\mathbf{s}$, where the vectors **c** and **s** are in the relation $\mathbf{c} = \boldsymbol{\Delta}\boldsymbol{\varphi}_2\boldsymbol{\Delta}'\mathbf{s}$ $(= \mathbf{N}_0\mathbf{k}^{-\delta}\mathbf{N}_0'\mathbf{s})$, and **s** satisfies the condition

$$[\mathbf{K}_0 - \mathbf{N}_0'(\mathbf{N}_0\mathbf{k}^{-\delta}\mathbf{N}_0')^{-}\mathbf{N}_0]\mathbf{N}_0'\mathbf{s} = \mathbf{0}, \tag{3.16}$$

or the equivalent condition

$$[\mathbf{K}_0 - \mathbf{N}_0'\mathbf{r}^{-\delta}\mathbf{N}_0(\mathbf{N}_0'\mathbf{r}^{-\delta}\mathbf{N}_0)^{-}\mathbf{K}_0]\mathbf{N}_0'\mathbf{s} = \mathbf{0}, \tag{3.17}$$

where $\mathbf{K}_0 = \mathbf{k}^{\delta} - n^{-1}\mathbf{k}\mathbf{k}'$ and $\mathbf{N}_0 = \mathbf{N} - n^{-1}\mathbf{r}\mathbf{k}'$ ($\equiv$ **A** in Pearce, 1983, p.97).

Proof. Under (3.4), with (3.14) and (3.15), the necessary and sufficient condition for a function $\mathbf{w}'\mathbf{y}_2 = \mathbf{w}'\boldsymbol{\varphi}_2\mathbf{y}$ to be the BLUE of $\mathrm{E}(\mathbf{w}'\mathbf{y}_2) = \mathbf{w}'\boldsymbol{\varphi}_2\boldsymbol{\Delta}'\boldsymbol{\tau}$ is, on account of Theorem 3 of Zyskind (1967), the equality

$$(\mathbf{I} - \mathbf{P}_{\boldsymbol{\varphi}_2\boldsymbol{\Delta}'})[\boldsymbol{\varphi}_2\mathbf{D}'\mathbf{D}\boldsymbol{\varphi}_2(\sigma_B^2 - K_H^{-1}\sigma_U^2) + \boldsymbol{\varphi}_2(\sigma_U^2 + \sigma_e^2)]\mathbf{w} = \mathbf{0}.$$

It holds uniformly if and only if the equalities

$$(\mathbf{I} - \mathbf{P}_{\boldsymbol{\varphi}_2\boldsymbol{\Delta}'})\boldsymbol{\varphi}_2\mathbf{w} = \mathbf{0} \text{ and } (\mathbf{I} - \mathbf{P}_{\boldsymbol{\varphi}_2\boldsymbol{\Delta}'})\boldsymbol{\varphi}_2\mathbf{D}'\mathbf{D}\boldsymbol{\varphi}_2\mathbf{w} = \mathbf{0}$$

hold simultaneously. The first equality holds if and only if $\boldsymbol{\varphi}_2\mathbf{w} = \boldsymbol{\varphi}_2\boldsymbol{\Delta}'\mathbf{s}$ for some **s**, which holds if and only if $\mathbf{D}\boldsymbol{\varphi}_2\mathbf{w} = \mathbf{D}\boldsymbol{\varphi}_2\boldsymbol{\Delta}'\mathbf{s}$ for that **s**. With this, the second equality reads

$$(\mathbf{I}_n - \mathbf{P}_{\boldsymbol{\varphi}_2\boldsymbol{\Delta}'})\boldsymbol{\varphi}_2\mathbf{D}'\mathbf{D}\boldsymbol{\varphi}_2\boldsymbol{\Delta}'\mathbf{s} = \mathbf{0},$$

which is equivalent to (3.16) due to the relations $\mathbf{D}'\mathbf{k}^{-\delta}\mathbf{D}\boldsymbol{\varphi}_2 = \boldsymbol{\varphi}_2$, $\mathbf{D}\boldsymbol{\varphi}_2\mathbf{D}' = \mathbf{K}_0$, $\boldsymbol{\Delta}\boldsymbol{\varphi}_2\mathbf{D}' = \mathbf{N}_0$ and $\boldsymbol{\Delta}\boldsymbol{\varphi}_2\boldsymbol{\Delta}' = \mathbf{N}_0\mathbf{k}^{-\delta}\mathbf{N}_0'$. That the conditions (3.16) and (3.17) are equivalent can be checked by utilizing the equality $\mathbf{N}_0'(\mathbf{N}_0\mathbf{k}^{-\delta}\mathbf{N}_0')^{-}\mathbf{N}_0\mathbf{k}^{-\delta}\mathbf{N}_0' = \mathbf{N}_0' = \mathbf{N}_0'\mathbf{r}^{-\delta}\mathbf{N}_0(\mathbf{N}_0'\mathbf{r}^{-\delta}\mathbf{N}_0)^{-}\mathbf{N}_0'$, obtainable from Lemma 2.2.6(c) of Rao and Mitra (1971), and by noting that $\mathbf{N}_0\mathbf{k}^{-\delta}\mathbf{K}_0 = \mathbf{N}_0$. Finally, the relation between **c** and **s** follows from the fact that $\mathrm{E}(\mathbf{s}'\boldsymbol{\Delta}\,\mathbf{y}_2) = \mathbf{s}'\mathbf{N}_0\mathbf{k}^{-\delta}\mathbf{N}_0'\boldsymbol{\tau}$. □

Corollary 3.1. For the estimation of $\mathbf{c}'\boldsymbol{\tau} = \mathbf{s}'\mathbf{N}_0\mathbf{k}^{-\delta}\mathbf{N}_0'\boldsymbol{\tau}$ under (3.4) the following applies:

(a) The case $\mathbf{N}_0'\mathbf{s} = \mathbf{0}$ is to be excluded.

(b) If $\mathbf{N}_0'\mathbf{s} \neq \mathbf{0}$, then $\mathbf{c}'\boldsymbol{\tau}$ is a contrast, and to satisfy (3.16) or (3.17) by the vector **s** it is necessary and sufficient that $\mathbf{K}_0\mathbf{N}_0'\mathbf{s} \in \mathcal{C}(\mathbf{N}_0'\mathbf{r}^{-\delta}\mathbf{N}_0) = \mathcal{C}(\mathbf{N}_0')$.

(c) If **s** is such that $\mathbf{r}'\mathbf{s} = \mathbf{0}$, then the conditions (3.16) and (3.17) can be replaced by

$$\mathbf{K}_0\mathbf{N}'\mathbf{s} = \mathbf{N}_0'(\mathbf{N}_0\mathbf{k}^{-\delta}\mathbf{N}_0')^{-}\mathbf{N}_0\mathbf{N}'\mathbf{s} \tag{3.18}$$

and

$$\mathbf{K}_0\mathbf{N}'\mathbf{s} = \mathbf{N}_0'\mathbf{r}^{-\delta}\mathbf{N}_0(\mathbf{N}_0'\mathbf{r}^{-\delta}\mathbf{N}_0)^{-}\mathbf{K}_0\mathbf{N}'\mathbf{s}, \tag{3.19}$$

respectively. To satisfy any of them it is then necessary and sufficient that $\mathbf{K}_0\mathbf{N}'\mathbf{s} \in \mathcal{C}(\mathbf{N}_0')$.

(d) The condition (ii) of Theorem 2.2 is sufficient for satisfying the equality (3.19), and hence (3.18), by any vector **s**, thus being sufficient for the equalities (3.16) and (3.17) to be satisfied by any **s**.

Proof. The result (a) is obvious, as $\mathbf{N}_0'\mathbf{s} = \mathbf{0}$ implies $\mathbf{c} = \mathbf{0}$. To prove (b) note that $\mathbf{N}_0'\mathbf{1}_v = \mathbf{0}$ and that the equation $\mathbf{N}_0'\mathbf{r}^{-\delta}\mathbf{N}_0\mathbf{x} = \mathbf{K}_0\mathbf{N}_0'\mathbf{s}$ is consistent if and only if (3.17) holds [see Theorem 2.3.1(d) of Rao and Mitra, 1971]. Alternatively, note that the equation $\mathbf{N}_0'\mathbf{x} = \mathbf{K}_0\mathbf{N}_0'\mathbf{s}$ is consistent if and only if (3.16) holds, since $(\mathbf{N}_0\mathbf{k}^{-\delta}\mathbf{N}_0')^{-}\mathbf{N}_0\mathbf{k}^{-\delta}$ can be used as a g-inverse of $\mathbf{N}_0'$. The result (c) is obvious, as $\mathbf{N}_0'\mathbf{s} = \mathbf{N}'\mathbf{s}$ if $\mathbf{r}'\mathbf{s} = 0$. The result (d) can easily be checked by using Lemma 2.2.6(c) of Rao and Mitra (1971) and the relation $\mathbf{K}_0\mathbf{N}' = \mathbf{N}_0'\widetilde{\mathbf{k}}^{\delta}$, held under the condition (ii) of Theorem 2.2, with the matrix $\widetilde{\mathbf{k}}^{\delta}$ defined as in Section 1, and further by noting that $\mathbf{N}_0' = \mathbf{N}'(\mathbf{I} - n^{-1}\mathbf{1}_v\mathbf{r}')$. □

Now, it may be noted that if the conditions of Theorem 3.2 are satisfied, then $\widehat{\mathbf{c}'\boldsymbol{\tau}} = \mathbf{s}'\boldsymbol{\Delta}\mathbf{y}_2$ is the BLUE of the contrast $\mathbf{c}'\boldsymbol{\tau}$ under (3.4), and that its variance has the form

$$\mathrm{Var}(\widehat{\mathbf{c}'\boldsymbol{\tau}}) = \mathbf{s}'\mathbf{N}_0\mathbf{N}_0'\mathbf{s}(\sigma_B^2 - K_H^{-1}\sigma_U^2) + \mathbf{s}'\mathbf{N}_0\mathbf{k}^{-\delta}\mathbf{N}_0'\mathbf{s}(\sigma_U^2 + \sigma_e^2). \tag{3.20}$$

Evidently, if $k_1 = k_2 = \ldots = k_b = k$, the variance (3.20) reduces to

$$\mathrm{Var}(\widehat{\mathbf{c}'\boldsymbol{\tau}}) = k^{-1}\mathbf{s}'\mathbf{N}_0\mathbf{N}_0'\mathbf{s}[k\,\sigma_B^2 + (1 - K_H^{-1}k)\,\sigma_U^2 + \sigma_e^2], \tag{3.21}$$

or alternatively to

$$\mathrm{Var}(\widehat{\mathbf{c}'\boldsymbol{\tau}}) = \mathbf{c}'(k^{-1}\mathbf{N}_0\mathbf{N}_0')^{-}\mathbf{c}\,[k\,\sigma_B^2 + (1 - K_H^{-1}k)\,\sigma_U^2 + \sigma_e^2],$$

which coincides with the formula given by Pearce (1983, p. 80) when $k = K_H$ and the technical error is not taken into account.

Remark 3.3. For vectors **s** such that $\mathbf{r}'\mathbf{s} = \mathbf{0}$, the condition of Theorem 3.2 is less restrictive than that of Theorem 2.1, as any such **s** satisfying (2.13) satisfies also the condition (3.19), while that satisfying (3.19) must not necessarily satisfy (2.13).

An answer to the question what is necessary and sufficient for the condition of Theorem 3.2 to be satisfied by any **s** can be given as follows.

Corollary 3.2. The condition (3.16) holds for any **s**, i.e. the equality

$$\mathbf{K}_0\mathbf{N}_0' = \mathbf{N}_0'(\mathbf{N}_0\mathbf{k}^{-\delta}\mathbf{N}_0')^{-}\mathbf{N}_0\mathbf{N}_0' \tag{3.22}$$

holds, if and only if for any $b\times 1$ vector $\mathbf{t}$ that satisfies the equality $\mathbf{N}_0\mathbf{t}=\mathbf{0}$ the equality $\mathbf{N}_0\mathbf{K}_0\mathbf{t}=\mathbf{0}$ holds as well.

Proof. From Corollary 3.1(b), the condition (3.16) is satisfied by any $\mathbf{s}$ if and only if $\mathcal{C}(\mathbf{K}_0\mathbf{N}_0')\subset\mathcal{C}(\mathbf{N}_0')$. But this inclusion holds if and only if $\mathcal{C}^{\perp}(\mathbf{N}_0')\subset\mathcal{C}^{\perp}(\mathbf{K}_0\mathbf{N}_0')$ or, equivalently, $\mathcal{N}(\mathbf{N}_0)\subset\mathcal{N}(\mathbf{N}_0\mathbf{K}_0)$ [since $\mathcal{C}^{\perp}(\mathbf{A})=\mathcal{N}(\mathbf{A}')$ for any matrix $\mathbf{A}$, i.e. the orthogonal complement of the column space of a matrix $\mathbf{A}$ is the null space of the transpose of $\mathbf{A}$; see, e.g., Seber, 1980, Lemma 1.2.1]. Evidently, the last inclusion means that for any vector $\mathbf{t}$ for which $\mathbf{N}_0\mathbf{t}=\mathbf{0}$ the equality $\mathbf{N}_0\mathbf{K}_0\mathbf{t}=\mathbf{0}$ also holds. ❑

To see the applicability of Corollary 3.2, it will be helpful to examine the following two examples taken from Pearce (1983, p.102 and p.117).

Example 3.1. Consider a design with the incidence matrix

$$\mathbf{N}=\begin{bmatrix}1&1&1&1&1&1\\1&1&1&1&1&1\\1&1&1&1&1&1\\1&1&1&1&1&1\\0&1&0&1&0&1\\0&0&1&1&1&0\end{bmatrix}.$$

From it,

$$\mathbf{N}_0=\frac{1}{10}\begin{bmatrix}2&0&-2&0&-2&2\\2&0&-2&0&-2&2\\2&0&-2&0&-2&2\\2&0&-2&0&-2&2\\-4&5&4&-5&4&-4\\-4&-5&4&5&4&-4\end{bmatrix}$$

and

$$\mathbf{N}_0\mathbf{K}_0=\frac{1}{150}\begin{bmatrix}136&20&-156&20&-156&136\\136&20&-156&20&-156&136\\136&20&-156&20&-156&136\\136&20&-156&20&-156&136\\-272&335&312&-415&312&-272\\-272&335&312&-415&312&-272\end{bmatrix}.$$

Taking, e.g., $\mathbf{t}=[-1,2,-1,2,-1-1]'$ it can be seen that $\mathbf{N}_0\mathbf{t}=[0,0,0,0,0,0]'$, while $\mathbf{N}_0\mathbf{K}_0\mathbf{t}=10^{-1}[8,8,8,8,-16,-16]'$. Thus, this design does not satisfy the condition of Corollary 3.2, and hence (3.16) does not hold for any $\mathbf{s}$.

Example 3.2. Consider a design with

$$\mathbf{N} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

from which

$$\mathbf{N}_0 = \frac{1}{14}\begin{bmatrix} 6 & 6 & -6 & -6 \\ 6 & 6 & -6 & -6 \\ 6 & 6 & -6 & -6 \\ 2 & -12 & 5 & 5 \\ -12 & 2 & 5 & 5 \\ -8 & -8 & 8 & 8 \end{bmatrix} \quad \text{and} \quad \mathbf{N}_0\mathbf{K}_0 = \frac{2}{7^2}\begin{bmatrix} 36 & 36 & -36 & -36 \\ 36 & 36 & -36 & -36 \\ 36 & 36 & -36 & -36 \\ 19 & 79 & 30 & 30 \\ 79 & 19 & 30 & 30 \\ -48 & -48 & 48 & 48 \end{bmatrix}.$$

The matrix $\mathbf{N}_0$ above is of rank 2, and hence the dimension of the null space of $\mathbf{N}_0$ is 2. From definitions, $\mathbf{N}_0\mathbf{1}_b = \mathbf{0}$ and $\mathbf{N}_0\mathbf{K}_0\mathbf{1}_b = \mathbf{0}$. Also, taking $\mathbf{t} = [0,0,-1,1]'$ one obtains both $\mathbf{N}_0\mathbf{t} = \mathbf{0}$ and $\mathbf{N}_0\mathbf{K}_0\mathbf{t} = \mathbf{0}$, which shows that the condition of Corollary 3.2 is satisfied. Thus, in this example the equality (3.16) holds for any **s**. This implies that the design provides under the model (3.4) the BLUE for any contrast $\mathbf{c}'\boldsymbol{\tau} = \mathbf{s}'\mathbf{N}_0\mathbf{k}^{-\delta}\mathbf{N}_0'\boldsymbol{\tau}$, i.e. for any $\mathbf{c}'\boldsymbol{\tau}$ such that $\mathbf{c} \in \mathcal{C}(\mathbf{N}_0\mathbf{k}^{-\delta}\mathbf{N}_0')$. Here

$$\mathbf{N}_0\mathbf{k}^{-\delta}\mathbf{N}_0' = \frac{1}{84}\begin{bmatrix} 18 & 18 & 18 & -15 & -15 & -24 \\ 18 & 18 & 18 & -15 & -15 & -24 \\ 18 & 18 & 18 & -15 & -15 & -24 \\ -15 & -15 & -15 & 23 & 2 & 20 \\ -15 & -15 & -15 & 2 & 23 & 20 \\ -24 & -24 & -24 & 20 & 20 & 32 \end{bmatrix},$$

and so the columns of this matrix span the subspace of all contrasts (of the vectors **c** representing them) for which the BLUEs under the inter-block model exist. The dimension of this subspace is 2. The reader can check that also the equivalent condition (3.17) holds for any **s** in this example.

Remark 3.4. (a) Since $\mathbf{N}_0\mathbf{1}_b = \mathbf{0}$ and $\mathbf{N}_0\mathbf{K}_0\mathbf{1}_b = \mathbf{0}$ always, the necessary and sufficient condition for the equality (3.22) can be replaced by the condition that $\mathbf{N}\mathbf{t}_0 = \mathbf{0}$ implies $\mathbf{N}_0\mathbf{k}^{\delta}\mathbf{t}_0 = \mathbf{0}$ for any vector $\mathbf{t}_0$ being $\mathbf{k}^{\delta}$-orthogonal to $\mathbf{1}_b$, i.e. such that $\mathbf{k}'\mathbf{t}_0 = \mathbf{0}$. [Thus, the condition (6.10) of Theorem 4 given by Kala (1991) is sufficient for the equality (3.22) to hold, but it is not necessary for that, as it has been established for the projection $\mathbf{P}_{\mathbf{D}'}\mathbf{y}$, not for (3.4).]

(b) If rank $(\mathbf{N}_0) = b-1$, i.e., the columns of the incidence matrix **N** are all linearly independent, then a vector **t** that satisfies $\mathbf{N}_0\mathbf{t} = \mathbf{0}$ must be equal or proportional to $\mathbf{1}_b$ [i.e., $\mathbf{t} \in \mathcal{C}(\mathbf{1}_b)$], and so satisfy also the equality $\mathbf{N}_0\mathbf{K}_0\mathbf{t} = \mathbf{0}$. Thus, the condition of Corollary 3.2 is then satisfied automatically, whatever the k_j 's are.

Example 3.3. Consider a design (from Pearce, 1983, p. 225) for a 2^3 factorial structure of treatments, with the incidence matrix

$$\mathbf{N}=\begin{bmatrix} 0&1&1&1&0&0&0&0\\ 0&0&0&0&2&1&1&1\\ 0&0&0&0&1&2&1&1\\ 0&0&0&0&1&1&2&1\\ 1&1&1&0&0&0&0&0\\ 1&1&0&1&0&0&0&0\\ 1&0&1&1&0&0&0&0\\ 0&0&0&0&1&1&1&2 \end{bmatrix}.$$

It is a non-binary, non-proper, non-equireplicate disconnected block design. It can be seen that the rank of **N** is equal here to the number of blocks (and of treatments, i.e., **N** is nonsingular), and so the condition of Corollary 3.2 is satisfied on account of Remark 3.4(b). The reader may note that the equality (3.22) holds here also on account of Corollary 3.1(d).

Remark 3.5. If the equation (3.16) holds for any s, i.e., if (3.22) holds, then

$$\text{Cov}(\mathbf{y}_2)\boldsymbol{\varphi}_2\boldsymbol{\Delta}'=\boldsymbol{\varphi}_2\boldsymbol{\Delta}'[(\mathbf{N}_0\mathbf{k}^{-\delta}\mathbf{N}_0')^{-}\mathbf{N}_0\mathbf{N}_0'(\sigma_B^2-K_H^{-1}\sigma_U^2)+\mathbf{I}_n(\sigma_U^2+\sigma_e^2)],$$

which implies that both $\boldsymbol{\varphi}_2\boldsymbol{\Delta}'\mathbf{s}$ and $\text{Cov}(\mathbf{y}_2)\boldsymbol{\varphi}_2\boldsymbol{\Delta}'\mathbf{s}$ belong to $\mathcal{C}(\boldsymbol{\varphi}_2\boldsymbol{\Delta}')$ for any s, and thus the condition stated in Theorem 4 of Zyskind (1967), when applied to Theorem 3.2 above, is satisfied.This means that the BLUEs obtainable under the submodel (3.4), with the moments (3.14) and (3.15), when the condition (3.22) is satisfied, can equivalently be obtained under a simple alternative model in which, instead of (3.15), the identity matrix $\mathbf{I}_n$ multiplied by a positive scalar is used as the covariance matrix of $\mathbf{y}_2$. Moreover, it can be shown (applying, e.g., Theorem 2.3.2 of Rao and Mitra, 1971) that the equality (3.22) is not only a sufficient but also a necessary condition for the SLSEs and the BLUEs to be the same.

Remark 3.5 implies, in particular, that if the equality (3.22) holds, then the BLUE of the expectation vector (3.14) is obtainable by a simple least squares procedure, i.e. has the form

$$\hat{E}(\mathbf{y}_2)=\mathbf{P}_{\varphi_2\Delta'}\mathbf{y}_2,$$

where$\mathbf{P}_{\varphi_2\Delta'}=\boldsymbol{\varphi}_2\boldsymbol{\Delta}'(\boldsymbol{\Delta}\boldsymbol{\varphi}_2\boldsymbol{\Delta}')^{-}\boldsymbol{\Delta}\boldsymbol{\varphi}_2$ Thus, the vector $\mathbf{y}_2$ can be decomposed as

$$\mathbf{y}_2=\mathbf{P}_{\varphi_2\Delta'}\mathbf{y}_2+(\mathbf{I}_n-\mathbf{P}_{\varphi_2\Delta'})\mathbf{y}_2 \quad \text{(in terms of } \mathbf{y}_2)$$

$$=\mathbf{P}_{\varphi_2\Delta'}\,\mathbf{y}+(\boldsymbol{\varphi}_2-\mathbf{P}_{\varphi_2\Delta'})\mathbf{y} \quad \text{(in terms of } \mathbf{y}).$$

The above decomposition yields the inter-block analysis of variance, of the form

$$\|\mathbf{y}_2\|^2=\|\mathbf{P}_{\varphi_2\Delta'}\mathbf{y}_2\|^2+\|(\mathbf{I}_n-\mathbf{P}_{\varphi_2\Delta'})\mathbf{y}_2\|^2,$$

expressible in terms of the observed vector **y** as

$$\mathbf{y}'\boldsymbol{\varphi}_2\mathbf{y}=\mathbf{y}'\boldsymbol{\varphi}_2\boldsymbol{\Delta}'(\boldsymbol{\Delta}\boldsymbol{\varphi}_2\boldsymbol{\Delta}')^{-}\boldsymbol{\Delta}\boldsymbol{\varphi}_2\mathbf{y}+\mathbf{y}'[\boldsymbol{\varphi}_2-\boldsymbol{\varphi}_2\boldsymbol{\Delta}'(\boldsymbol{\Delta}\boldsymbol{\varphi}_2\boldsymbol{\Delta}')^{-}\boldsymbol{\Delta}\boldsymbol{\varphi}_2]\mathbf{y}=\mathbf{Q}_2'\mathbf{C}_2^{-}\mathbf{Q}_2+\mathbf{y}'\boldsymbol{\psi}_2\mathbf{y},$$

where $\mathbf{Q}_2 = \Delta\boldsymbol{\varphi}_2\mathbf{y}$, $\mathbf{C}_2 = \Delta\boldsymbol{\varphi}_2\Delta'$ and $\boldsymbol{\psi}_2 = \boldsymbol{\varphi}_2 - \boldsymbol{\varphi}_2\Delta'\mathbf{C}_2^-\Delta\boldsymbol{\varphi}_2 = \boldsymbol{\varphi}_2(\mathbf{I}_n - \Delta'\mathbf{C}_2^-\Delta)\boldsymbol{\varphi}_2$ (i.e., as $\mathbf{Q}_0$, $\mathbf{C}_0$ and $\boldsymbol{\psi}_0$ in Pearce, 1983, Section 3.8). The quadratic form $\mathbf{y}'\boldsymbol{\varphi}_2\mathbf{y}$ can be called the inter-block total sum of squares, while its components, $\mathbf{Q}_2'\mathbf{C}_2^-\mathbf{Q}_2$ and $\mathbf{y}'\boldsymbol{\psi}_2\mathbf{y}$, can be called the inter-block treatment sum of squares and the inter-block residual sum of squares, respectively. The corresponding d.f. are evidently $b-1=\text{rank}(\boldsymbol{\varphi}_2)$ for the total, $v-\rho-1=\text{rank}(\mathbf{C}_2)$ for the treatment component ($v-\rho$ being the rank of $\mathbf{N}$) and $b-v+\rho=\text{rank}(\boldsymbol{\psi}_2)=\text{rank}(\boldsymbol{\varphi}_2)-\text{rank}(\mathbf{C}_2)$ for the residual component. It is evident that the two component sums of squares are distributed independently. Their expectations are

$$\text{E}(\mathbf{Q}_2'\mathbf{C}_2^-\mathbf{Q}_2) = \text{tr}[\mathbf{N}_0'(\mathbf{N}_0\mathbf{k}^{-\delta}\mathbf{N}_0')^-\mathbf{N}_0](\sigma_B^2 - K_H^{-1}\sigma_U^2) + (v-1-\rho)(\sigma_U^2+\sigma_e^2) + \boldsymbol{\tau}'\mathbf{C}_2\boldsymbol{\tau}$$

and

$$\text{E}(\mathbf{y}'\boldsymbol{\psi}_2\mathbf{y}) = \{\text{tr}(\mathbf{K}_0) - \text{tr}[\mathbf{N}_0'(\mathbf{N}_0\mathbf{k}^{-\delta}\mathbf{N}_0')^-\mathbf{N}_0]\}\,(\sigma_B^2 - K_H^{-1}\sigma_U^2) + (b-v+\rho)(\sigma_U^2+\sigma_e^2).$$

It follows from the latter equality that the inter-block residual mean square $s_2^2 = \mathbf{y}'\boldsymbol{\varphi}_2\mathbf{y}/(b-v+\rho)$ is an unbiased estimator, the MINQUE in fact, of

$$\sigma_2^2 = (b-v+\rho)^{-1}\{\text{tr}(\mathbf{K}_0) - \text{tr}[\mathbf{N}_0'(\mathbf{N}_0\mathbf{k}^{-\delta}\mathbf{N}_0')^-\mathbf{N}_0]\}\,(\sigma_B^2 - K_H^{-1}\sigma_U^2) + \sigma_U^2 + \sigma_e^2. \qquad (3.23)$$

In case of $k_1 = k_2 = \ldots = k_b = k$,

$$\sigma_2^2 = k\,\sigma_B^2 + (1 - K_H^{-1}k)\,\sigma_U^2 + \sigma_e^2,$$

further reducing to $k\,\sigma_B^2 + \sigma_e^2$, if $k = K_H$. It should be noted, however, that $b-v+\rho = 0$ if $b = v-\rho$ (obviously $b \geq v-\rho$ always). In that case no estimator for σ_2^2 exists in the inter-block analysis.

Thus, in the case of equal k_j's and $b > v-\rho$, the mean square s_2^2 can be used to obtain an unbiased estimator of the variance (3.21), in the form

$$\hat{\text{V}}\text{ar}(\mathbf{c}'\hat{\boldsymbol{\tau}}) = k^{-1}\mathbf{s}'\mathbf{N}_0\mathbf{N}_0'\mathbf{s}s_2^2 = k\,\mathbf{c}'(\mathbf{N}_0\mathbf{N}_0')^-\mathbf{c}s_2^2.$$

In general, the estimation of (3.21) is not so simple.

Furthermore, if $k_1 = k_2 = \ldots = k_b = k$, then $\text{Cov}(\mathbf{y}_2) = \boldsymbol{\varphi}_2\sigma_2^2$ and, similarly as for the intra-block analysis, it can be shown that, under the multivariate normality assumption, both $\mathbf{Q}_2'\mathbf{C}_2^-\mathbf{Q}_2/\sigma_2^2$ and $\mathbf{y}'\boldsymbol{\psi}_2\mathbf{y}/\sigma_2^2$ have χ^2 distributions, the first on $v-\rho-1$ d.f. with the non-centrality parameter $\delta = \boldsymbol{\tau}'\mathbf{C}_2\boldsymbol{\tau}/\sigma_2^2$, the second on $b-v+\rho$ d.f. with $\delta = 0$. Hence, the hypothesis $\boldsymbol{\tau}'\mathbf{C}_2\boldsymbol{\tau} = 0$, equivalent to $\text{E}(\mathbf{y}_2) = \mathbf{0}$ [or $\mathbf{P}_{\mathbf{D}'}\,\text{E}(\mathbf{y}) \in \mathcal{C}(\mathbf{1}_n)$], can be tested by the variance ratio criterion

$$(v-\rho-1)^{-1}\mathbf{Q}_2'\mathbf{C}_2^-\mathbf{Q}_2/s_2^2,$$

which under the assumed normality has then the F distribution with $v-\rho-1$ d.f., central when the hypothesis is true. This, however, does not apply to the general case, when k_j 's are not all equal.

3.3. The total-area submodel

Considering the third submodel, (3.6), it is evident that its properties are

$$\mathrm{E}(\mathbf{y}_3) = \boldsymbol{\varphi}_3\boldsymbol{\Delta}'\boldsymbol{\tau} = n^{-1}\mathbf{1}_n\mathbf{r}'\boldsymbol{\tau} \tag{3.24}$$

and

$$\mathrm{Cov}(\mathbf{y}_3) = \boldsymbol{\varphi}_3[(n^{-1}\mathbf{k}'\mathbf{k} - N_B^{-1}n)\sigma_B^2 + (1-K_H^{-1}n^{-1}\mathbf{k}'\mathbf{k})\sigma_U^2 + \sigma_e^2]. \tag{3.25}$$

They lead to the following main result concerning estimation under (3.6).

Theorem 3.3. Under (3.6), a function $\mathbf{w}'\mathbf{y}_3 = \mathbf{w}'\boldsymbol{\varphi}_3\mathbf{y}$ is uniformly the BLUE of $\mathbf{c}'\boldsymbol{\tau}$ if and only if $\boldsymbol{\varphi}_3\mathbf{w} = \boldsymbol{\varphi}_3\boldsymbol{\Delta}'\mathbf{s}$, where the vectors $\mathbf{c}$ and $\mathbf{s}$ are in the relation $\mathbf{c} = \boldsymbol{\Delta}\boldsymbol{\varphi}_3\boldsymbol{\Delta}'\mathbf{s}$ $(= n^{-1}\mathbf{r}\mathbf{r}'\mathbf{s})$.

Proof. It follows exactly the same pattern as the proof of Theorem 3.1. ❑

Remark 3.6.(a) The only parametric functions for which the BLUEs under (3.6) exist are those defined as $\mathbf{c}'\boldsymbol{\tau} = (\mathbf{s}'\mathbf{r})n^{-1}\mathbf{r}'\boldsymbol{\tau}$, i.e. the general parametric mean and its multiplicities, contrasts being excluded a fortiori (as $\mathbf{1}_v'\mathbf{c} = \mathbf{r}'\mathbf{s}$ here).

(b) Since $\mathrm{Cov}(\mathbf{y}_3)\boldsymbol{\varphi}_3\boldsymbol{\Delta}' = \boldsymbol{\varphi}_3\boldsymbol{\Delta}'[(n^{-1}\mathbf{k}'\mathbf{k} - N_B^{-1}n)\sigma_B^2 + (1-K_H^{-1}n^{-1}\mathbf{k}'\mathbf{k})\sigma_U^2 + \sigma_e^2]$, the BLUEs under (3.6) and the SLSEs are the same (on account of Zyskind's, 1967, Theorem 4 applied to Theorem 3.3).

If $\mathbf{c}'\boldsymbol{\tau} = (\mathbf{s}'\mathbf{r})n^{-1}\mathbf{r}'\boldsymbol{\tau} = (\mathbf{c}'\mathbf{1}_v)n^{-1}\mathbf{r}'\boldsymbol{\tau}$, then the variance of its BLUE under (3.6), i.e. of $\widehat{\mathbf{c}'\boldsymbol{\tau}} = \mathbf{s}'\boldsymbol{\Delta}\,\mathbf{y}_3$, is of the form

$$\begin{aligned}\mathrm{Var}(\widehat{\mathbf{c}'\boldsymbol{\tau}}) &= \mathbf{s}'\boldsymbol{\Delta}\boldsymbol{\varphi}_3\boldsymbol{\Delta}'\mathbf{s}[(n^{-1}\mathbf{k}'\mathbf{k} - N_B^{-1}n)\sigma_B^2 + (1-K_H^{-1}n^{-1}\mathbf{k}'\mathbf{k})\sigma_U^2 + \sigma_e^2] \\ &= n^{-1}(\mathbf{c}'\mathbf{1}_v)^2[(n^{-1}\mathbf{k}'\mathbf{k} - N_B^{-1}n)\sigma_B^2 + (1-K_H^{-1}n^{-1}\mathbf{k}'\mathbf{k})\sigma_U^2 + \sigma_e^2].\end{aligned} \tag{3.26}$$

Evidently, if all k_j are equal $(=k)$, the variance (3.26) reduces to

$$\mathrm{Var}(\widehat{\mathbf{c}'\boldsymbol{\tau}}) = n^{-1}(\mathbf{c}'\mathbf{1}_v)^2[(1-N_B^{-1}b)k\sigma_B^2 + (1-K_H^{-1}k)\sigma_U^2 + \sigma_e^2]\ ,$$

and if, in addition, $b = N_B$ and $K_H = k$, which may be considered as the usual case, then $\mathrm{Var}(\widehat{\mathbf{c}'\boldsymbol{\tau}}) = n^{-1}(\mathbf{c}'\mathbf{1}_v)^2\sigma_e^2$.

Finally, it may be noted that since $\mathbf{P}_{\varphi_3\Delta'} = \boldsymbol{\varphi}_3$ (as $n^{-1}\mathbf{1}_v\mathbf{1}_v'$ is a g-inverse of $\boldsymbol{\Delta}\boldsymbol{\varphi}_3\boldsymbol{\Delta}' = n^{-1}\mathbf{r}\mathbf{r}'$) and, hence, both

$$(\mathbf{I}_n - \mathbf{P}_{\varphi_3\Delta'})\mathbf{y}_3 = \mathbf{0} \text{ and } (\mathbf{I}_n - \mathbf{P}_{\varphi_3\Delta'})\,\mathrm{Cov}(\mathbf{y}_3) = 0,$$

the vector $\mathbf{P}_{\varphi_3\Delta'}\,\mathbf{y}_3 = \mathbf{y}_3 = n^{-1}\mathbf{1}_n\mathbf{1}_n'\mathbf{y}$ is itself the BLUE of its expectation, $n^{-1}\mathbf{1}_n\mathbf{r}'\boldsymbol{\tau}$, leaving no residuals.

3.4. Some special cases

It follows from the considerations above that any function $\mathbf{s}'\boldsymbol{\Delta}\,\mathbf{y}$ can be resolved into three components in the form

$$\mathbf{s}'\boldsymbol{\Delta}\,\mathbf{y} = \mathbf{s}'\boldsymbol{\Delta}\,\mathbf{y}_1 + \mathbf{s}'\boldsymbol{\Delta}\mathbf{y}_2 + \mathbf{s}'\boldsymbol{\Delta}\,\mathbf{y}_3 = \mathbf{s}'\boldsymbol{\Delta}\,\boldsymbol{\varphi}_1\mathbf{y} + \mathbf{s}'\boldsymbol{\Delta}\,\boldsymbol{\varphi}_2\mathbf{y} + \mathbf{s}'\boldsymbol{\Delta}\,\boldsymbol{\varphi}_3\mathbf{y},$$

which conveniently can be written as

$$\mathbf{s}'\boldsymbol{\Delta}\,\mathbf{y} = \mathbf{s}'\mathbf{Q}_1 + \mathbf{s}'\mathbf{Q}_2 + \mathbf{s}'\mathbf{Q}_3, \tag{3.27}$$

with $\mathbf{Q}_1 = \boldsymbol{\Delta}\,\mathbf{y}_1 = \boldsymbol{\Delta}\,\boldsymbol{\varphi}_1\mathbf{y}$, $\mathbf{Q}_2 = \boldsymbol{\Delta}\,\mathbf{y}_2 = \boldsymbol{\Delta}\,\boldsymbol{\varphi}_2\mathbf{y}$ and $\mathbf{Q}_3 = \boldsymbol{\Delta}\,\mathbf{y}_3 = \boldsymbol{\Delta}\,\boldsymbol{\varphi}_3\mathbf{y}$. Each of the components in (3.27) represents a contribution to $\mathbf{s}'\boldsymbol{\Delta}\mathbf{y}$ from a different stratum. The component $\mathbf{s}'\mathbf{Q}_1$ may then be called the intra-block component, $\mathbf{s}'\mathbf{Q}_2$ the inter-block component and $\mathbf{s}'\mathbf{Q}_3$ the total-area component.

In connection with formula (3.27) it is interesting to consider three special cases of the vector $\mathbf{s}$ (and hence of $\mathbf{c} = \mathbf{r}^{\delta}\mathbf{s}$). First, suppose that $\mathbf{s}$ is such that $\mathbf{N}'\mathbf{s} = \mathbf{0}$, i.e., is orthogonal to the columns of $\mathbf{N}$, which also implies that $\mathbf{r}'\mathbf{s} = 0$ (i.e. $\mathbf{1}_v'\mathbf{c} = 0$). Then $\mathbf{s}'\boldsymbol{\Delta}\,\mathbf{y} = \mathbf{s}'\mathbf{Q}_1$, which means that only the intra-block stratum contributes. As the second case, suppose that $\mathbf{s}$ is such that $\mathbf{N}'\mathbf{s} \neq \mathbf{0}$ but it satisfies the conditions $\boldsymbol{\varphi}_1\boldsymbol{\Delta}'\,\mathbf{s} = \mathbf{0}$ and $\mathbf{r}'\mathbf{s} = 0$. Then $\mathbf{s}'\boldsymbol{\Delta}\,\mathbf{y} = \mathbf{s}'\mathbf{Q}_2$, which means that the contribution comes from the inter-block stratum only. As the third case suppose that $\mathbf{s} \in \mathcal{C}(\mathbf{1}_v)$, i.e., that $\mathbf{s}$ is proportional to the vector $\mathbf{1}_v$. Then, on account of (3.10), $\mathbf{s}'\boldsymbol{\Delta}\,\mathbf{y} = \mathbf{s}'\mathbf{Q}_3$, which means that the only contribution is from the total-area stratum.

Moreover, for the three cases it is instructive to observe the following. If $\mathbf{N}'\mathbf{s} = \mathbf{0}$, then the condition (2.13) is satisfied. Also, if $\mathbf{N}'\mathbf{s} \neq \mathbf{0}$ but $\boldsymbol{\varphi}_1\boldsymbol{\Delta}'\mathbf{s} = \mathbf{0}$, which is equivalent to $\boldsymbol{\Delta}\boldsymbol{\varphi}_1\boldsymbol{\Delta}'\mathbf{s} = \mathbf{0}$, the condition (2.13) holds, provided the condition (ii) of Theorem 2.2 holds. Finally, if $\mathbf{s} \in \mathcal{C}(\mathbf{1}_v)$, then $\mathbf{N}'\mathbf{s} \in \mathcal{C}(\mathbf{k})$ and (2.13) is satisfied, provided

$$\mathcal{C}(\mathbf{k}) \subset \mathcal{C}\{\mathrm{diag}[\mathbf{1}_{b_1} : \mathbf{1}_{b_2} : \ldots : \mathbf{1}_{b_g}]\},$$

i.e. again under the condition (ii) of Theorem 2.2. The importance of these observations is that for the three discussed cases the BLUE obtainable under the relevant submodel (3.2), (3.4) or (3.6), respectively , is simultaneously the BLUE of $\mathbf{s}'\mathbf{r}^{\delta}\boldsymbol{\tau}$ $(= \mathbf{c}'\boldsymbol{\tau})$ under the overall model (2.1), for the second and the third case, however, provided the design satisfies the condition (ii) of Theorem 2.2. (But see also Remark 3.7.)

The present discussion can be summarized as follows.

Corollary 3.3. The function $\mathbf{s}'\boldsymbol{\Delta}\,\mathbf{y}$ is the BLUE of $\mathbf{c}'\boldsymbol{\tau} = \mathbf{s}'\mathbf{r}^{\delta}\boldsymbol{\tau}$ under the overall model (2.1) in the following three cases:

(a) $\mathbf{N's} = \mathbf{0}$ (implying $\mathbf{r's} = 0$); the BLUE is then equal to $\mathbf{s'Q}_1$ and its variance is of the form

$$\mathrm{Var}(\mathbf{c}'\hat{\boldsymbol{\tau}}) = \mathbf{s'r}^{\delta}\mathbf{s}(\sigma_U^2 + \sigma_e^2) = \mathbf{c'r}^{-\delta}\mathbf{c}\,(\sigma_U^2 + \sigma_e^2). \tag{3.28}$$

(b) $\mathbf{N's} \neq \mathbf{0}$, but $\boldsymbol{\varphi}_1\boldsymbol{\Delta}'\mathbf{s} = \mathbf{0}$, and $\mathbf{r's} = 0$, provided the condition (ii) of Theorem 2.2 holds; the BLUE is then equal to $\mathbf{s'Q}_2$ and its variance is of the form

$$\mathrm{Var}(\mathbf{c}'\hat{\boldsymbol{\tau}}) = \mathbf{s'NN's}(\sigma_B^2 - K_H^{-1}\sigma_U^2) + \mathbf{s'r}^{\delta}\mathbf{s}\,(\sigma_U^2 + \sigma_e^2). \tag{3.29}$$

(c) $\mathbf{s} = s\,\mathbf{1}_v$, provided the condition (ii) of Theorem 2.2 holds; the BLUE is then equal to $\mathbf{s'Q}_3$ and its variance is of the form

$$\mathrm{Var}(\mathbf{c}'\hat{\boldsymbol{\tau}}) = s^2[(\mathbf{k'k} - N_B^{-1}n^2)\sigma_B^2 + (n - K_H^{-1}\mathbf{k'k})\,\sigma_U^2 + n\,\sigma_e^2]. \tag{3.30}$$

Proof. These results follow from the discussion above, the formulae (3.28), (3.29) and (3.30) being obtainable directly from (2.11), but also from (3.13), (3.20) and (3.26), respectively, in the last case by noting that $s = n^{-1}\mathbf{c'1}_v$.

Remark 3.7. For the case (b) of Corollary 3.3 it should be noticed that the conditions $\boldsymbol{\varphi}_1\boldsymbol{\Delta}'\mathbf{s} = \mathbf{0}$ and $\mathbf{r's} = 0$ imply that the design is disconnected, and hence (as can be shown), that

$$\mathbf{N's} = [\mathbf{s}_1'\mathbf{N}_1 : \mathbf{s}_2'\mathbf{N}_2 : \ldots : \mathbf{s}_g'\mathbf{N}_g]' = [s_1\mathbf{k}_1' : s_2\mathbf{k}_2' : \ldots : s_g\mathbf{k}_g']',$$

where $\mathbf{s}_l = s_l\mathbf{1}_{v_l}$ and $\mathbf{k}_l = \mathbf{N}_l'\mathbf{1}_{v_l}$. On the other hand, if the condition (2.13) is to be satisfied as well, then, on account of Corollary 2.1(b), it is necessary and sufficient that

$$\mathbf{k}_l = k_l\mathbf{1}_{b_l} \quad \text{for any } l \text{ such that } s_l \neq 0, \tag{3.31}$$

which in turn holds if and only if the block sizes of the design are constant within any of its connected subdesigns to which the nonzero s_l 's correspond, a condition that is weaker than (ii) of Theorem 2.2.

If (3.31) holds, then

$$\mathbf{s'NN's} = \sum_{l=1}^{g} s_l^2\mathbf{k}_l'\mathbf{k}_l = \sum_{l=1}^{g} s_l^2k_l^2b_l \geq \frac{(\sum_{l=1}^{g} s_l^2b_lk_l)^2}{\sum_{l=1}^{g} s_l^2b_l}, \tag{3.32}$$

the equality in (3.32) evidently holding if and only if the k_l 's involved by the nonzero s_l's are all equal (=k, say). In this extreme case formula (3.29) is reduced to

$$\mathrm{Var}(\mathbf{c}'\hat{\boldsymbol{\tau}}) = \mathbf{s'r}^{\delta}\mathbf{s}[k\,\sigma_B^2 + (1 - K_H^{-1}k)\,\sigma_U^2 + \sigma_e^2] = \mathbf{c'r}^{-\delta}\mathbf{c}\,[k\,\sigma_B^2 + (1 - K_H^{-1}k)\,\sigma_U^2 + \sigma_e^2]. \tag{3.33}$$

Remark 3.8. For the case (c) of Corollary 3.3 it can be noted that if $\mathbf{s} = s\mathbf{1}_v$, then $\mathbf{N}'\mathbf{s} = s\mathbf{k}$. Hence, on account of Corollary 2.1(b), the condition (ii) of Theorem 2.2 is not only sufficient but also necessary for the condition (2.13) to be satisfied in that case.

4. Estimating the same contrast under the intra-block and the inter-block submodels

The discussion in Section 3.4 has revealed that under certain conditions the function $\mathbf{s}'\mathbf{\Delta}\,\mathbf{y}$ is the BLUE of its expectation $\mathbf{c}'\boldsymbol{\tau} = \mathbf{s}'\mathbf{r}^{\delta}\boldsymbol{\tau}$, the decomposition (3.27) being then reduced to one component only. In particular, $\mathbf{s}'\mathbf{\Delta}\,\mathbf{y} = \mathbf{s}'\mathbf{Q}_3$ for any $\mathbf{s} \in \mathcal{C}(\mathbf{1}_v)$, as $\mathbf{1}_v'\mathbf{Q}_1 = \mathbf{1}_v'\mathbf{Q}_2 = 0$. On the other hand, for any $\mathbf{s}$ representing a contrast, i.e. for any $\mathbf{s}$ such that $\mathbf{s}'\mathbf{r} = 0$, the component $\mathbf{s}'\mathbf{Q}_3$ is equal to 0 and so (3.27) is reduced to

$$\mathbf{s}'\mathbf{\Delta}\,\mathbf{y} = \mathbf{s}'\mathbf{Q}_1 + \mathbf{s}'\mathbf{Q}_2\ , \tag{4.1}$$

unless some additional conditions for s are met which result in reducing (4.1) further, to one component only (Corollary 3.3).

The two components in (4.1) are uncorrelated, whatever the vector s and the matrix $\mathbf{\Delta}$ are, as there is no correlation between the vectors $\mathbf{y}_1$ and $\mathbf{y}_2$ in the decomposition (3.1). This is due to the equality $\boldsymbol{\varphi}_1\mathrm{Cov}(\mathbf{y})\boldsymbol{\varphi}_2 = \mathbf{0}$, holding on account of (2.11) and the properties of $\boldsymbol{\varphi}_1$ and $\boldsymbol{\varphi}_2$ shown in (3.8) and (3.10).

A question now arises under what condition the intra-block component $\mathbf{s}'\mathbf{Q}_1$ and the inter-block component $\mathbf{s}'\mathbf{Q}_2$ estimate the same parametric function, the contrast $\mathbf{c}'\boldsymbol{\tau} = \mathbf{s}'\mathbf{r}^{\delta}\boldsymbol{\tau}$ (with the accuracy to a constant factor). To answer this question note that

$$E(\mathbf{s}'\mathbf{Q}_1) = \mathbf{s}'\mathbf{\Delta}\boldsymbol{\varphi}_1\mathbf{\Delta}'\boldsymbol{\tau} \tag{4.2}$$

and

$$E(\mathbf{s}'\mathbf{Q}_2) = \mathbf{s}'\mathbf{\Delta}\boldsymbol{\varphi}_2\mathbf{\Delta}'\boldsymbol{\tau} = \mathbf{s}'\mathbf{\Delta}(\mathbf{I}_n - \boldsymbol{\varphi}_1)\mathbf{\Delta}'\boldsymbol{\tau}\,, \tag{4.3}$$

the last equality holding if $\mathbf{s}'\mathbf{r} = 0$. This implies the following.

Lemma 4.1. If s represents a contrast, then the equality $E(\mathbf{s}'\mathbf{Q}_1) = \kappa\, E(\mathbf{s}'\mathbf{Q}_2)$, where κ is a positive scalar, holds for any $\boldsymbol{\tau}$ if and only if

$$\mathbf{\Delta}\boldsymbol{\varphi}_1\mathbf{\Delta}'\mathbf{s} = \varepsilon\, \mathbf{r}^{\delta}\mathbf{s} \quad , \text{ with } \quad \varepsilon = \kappa/(1+\kappa)\ , \tag{4.4}$$

i.e., if and only if the vector s is an eigenvector of $\mathbf{C} = \mathbf{\Delta}\boldsymbol{\varphi}_1\mathbf{\Delta}'$ with respect to $\mathbf{r}^{\delta}$, corresponding to an eigenvalue ε such that $0 < \varepsilon < 1$.

This result was originally noticed by Jones (1959). Contrasts represented by eigenvectors of $\mathbf{C}$ with respect to $\mathbf{r}^{\delta}$ were later considered by Caliński (1971, 1977), and have been called basic contrasts by Pearce et al. (1974).

For any vector s satisfying the eigenvector condition (4.4), with $0 < \varepsilon < 1$, it follows from Theorem 3.1 that the component $\mathbf{s}'\mathbf{Q}_1$ is under (3.2) the BLUE of

$$E(\mathbf{s}'\mathbf{Q}_1) = \varepsilon\, \mathbf{s}'\mathbf{r}^{\delta}\boldsymbol{\tau}$$

and has the variance

$$\mathrm{Var}(\mathbf{s}'\mathbf{Q}_1) = \varepsilon\, \mathbf{s}'\mathbf{r}^{\delta}\mathbf{s}\, (\sigma_U^2 + \sigma_e^2), \tag{4.5}$$

while from Theorem 3.2 and Corollary 3.1(c), when (3.18) or (3.19) holds, the component $\mathbf{s}'\mathbf{Q}_2$ is under (3.4) the BLUE of

$$E(\mathbf{s}'\mathbf{Q}_2) = (1-\varepsilon)\mathbf{s}'\mathbf{r}^{\delta}\boldsymbol{\tau}$$

and has the variance

$$\mathrm{Var}(\mathbf{s}'\mathbf{Q}_2) = \mathbf{s}'\mathbf{N}\mathbf{N}'\mathbf{s}(\sigma_B^2 - K_H^{-1}\sigma_U^2) + (1-\varepsilon)\mathbf{s}'\mathbf{r}^{\delta}\mathbf{s}\, (\sigma_U^2 + \sigma_e^2)$$
$$= (1-\varepsilon)\mathbf{s}'\mathbf{r}^{\delta}\mathbf{s}\, [\mathbf{t}'\mathbf{k}^{2\delta}\mathbf{t}(\sigma_B^2 - K_H^{-1}\sigma_U^2) + \sigma_U^2 + \sigma_e^2], \tag{4.6}$$

where $\mathbf{t}$ is a $\mathbf{k}^{\delta}$-normalized eigenvector of $\mathbf{N}'\mathbf{r}^{-\delta}\mathbf{N}$ with respect to $\mathbf{k}^{\delta}$ corresponding to the eigenvalue $\mu = 1 - \varepsilon$ and related to the vector s by the formula

$$\mathbf{t} = (1-\varepsilon)^{-1/2}(\mathbf{s}'\mathbf{r}^{\delta}\mathbf{s})^{-1/2}\mathbf{k}^{-\delta}\mathbf{N}'\mathbf{s}, \tag{4.7}$$

and where $\mathbf{k}^{2\delta} = (\mathbf{k}^{\delta})^2$. Thus, the following theorem is established.

Theorem 4.1. For any $\mathbf{c} = \mathbf{r}^{\delta}\mathbf{s}$, where s is such that $\mathbf{1}_v'\mathbf{c} = \mathbf{r}'\mathbf{s} = 0$ and (4.4) is satisfied with $0 < \varepsilon < 1$, the BLUE of the contrast $\mathbf{c}'\boldsymbol{\tau}$ obtainable in the intra-block analysis can be written as

$$(\widehat{\mathbf{c}'\boldsymbol{\tau}})_{\mathrm{intra}} = \varepsilon^{-1}\mathbf{s}'\mathbf{Q}_1 = \varepsilon^{-1}\mathbf{c}'\mathbf{r}^{-\delta}\mathbf{Q}_1, \tag{4.8}$$

with

$$\mathrm{Var}[(\widehat{\mathbf{c}'\boldsymbol{\tau}})_{\mathrm{intra}}] = \varepsilon^{-1}\mathbf{c}'\mathbf{r}^{-\delta}\mathbf{c}\, (\sigma_U^2 + \sigma_e^2), \tag{4.9}$$

and that obtainable in the inter-block analysis, under the condition (3.18) or (3.19), as

$$(\widehat{\mathbf{c}'\boldsymbol{\tau}})_{\mathrm{inter}} = (1-\varepsilon)^{-1}\mathbf{s}'\mathbf{Q}_2 = (1-\varepsilon)^{-1}\mathbf{c}'\mathbf{r}^{-\delta}\mathbf{Q}_2, \tag{4.10}$$

with

$$\mathrm{Var}[(\widehat{\mathbf{c}'\boldsymbol{\tau}})_{\mathrm{inter}}] = (1-\varepsilon)^{-1}\mathbf{c}'\mathbf{r}^{-\delta}\mathbf{c}\, [\mathbf{t}'\mathbf{k}^{2\delta}\mathbf{t}(\sigma_B^2 - K_H^{-1}\sigma_U^2) + \sigma_U^2 + \sigma_e^2], \tag{4.11}$$

where $\mathbf{t}$ is as in (4.7). If, in particular, $k_1 = k_2 = \ldots = k_b = k$, then the variance (4.11) gets the form

$$\mathrm{Var}[(\widehat{\mathbf{c}'\boldsymbol{\tau}})_{\mathrm{inter}}] = (1-\varepsilon)^{-1}\mathbf{c}'\mathbf{r}^{-\delta}\mathbf{c}\, [k\,\sigma_B^2 + (1 - K_H^{-1}k)\,\sigma_U^2 + \sigma_e^2], \tag{4.12}$$

or

$$\mathrm{Var}[(\widehat{\mathbf{c}'\boldsymbol{\tau}})_{\mathrm{inter}}] = (1-\varepsilon)^{-1}\mathbf{c}'\mathbf{r}^{-\delta}\mathbf{c}\,(k\,\sigma_B^2+\sigma_e^2),$$

if in addition $k = K_H$.

Proof. It follows from the discussion preceding the theorem.

Remark 4.1. If ε is nonzero but less than 1, it is said that the contrast represented by **s**, i.e. $\mathbf{c}'\boldsymbol{\tau} = \mathbf{s}'\mathbf{r}^{\delta}\boldsymbol{\tau}$, is "partially confounded" with blocks. It would be said "totally confounded" in case of $\varepsilon = 0$. In the opposite case, when $\mathbf{N}'\mathbf{s} = \mathbf{0}$, which corresponds to $\varepsilon = 1$, one can say that the contrast is not confounded with blocks at all.

To give a statistical interpretation of the eigenvalue ε in (4.4), it will be instructive to suppose that, for the eigenvector **s** and the fixed vector **r** determining the contrast $\mathbf{c}'\boldsymbol{\tau} = \mathbf{s}'\mathbf{r}^{\delta}\boldsymbol{\tau}$ under study, there exists a design with the incidence matrix **N** such that $\mathbf{N}'\mathbf{s} = \mathbf{0}$. Then, according to Corollary 3.3(a), $\mathbf{s}'\mathbf{Q}_1 = \mathbf{s}'\boldsymbol{\Delta}\mathbf{y}$ is the BLUE of $\mathbf{c}'\boldsymbol{\tau} = \mathbf{s}'\mathbf{r}^{\delta}\boldsymbol{\tau}$ under the overall model (2.1), with the variance obtainable in the form (3.28). Comparing (3.28) with (4.9) it becomes evident that ε is the "efficiency factor" of the analysed design for the contrast $\mathbf{c}'\boldsymbol{\tau}$ when it is estimated in the intra-block analysis, i.e. by (4.8). On the other hand, $1-\varepsilon$ can be interpreted as the relative "loss of information" incurred in that analysis due to partially confounding the contrast with blocks. (This terminology is due to Jones, 1959, p. 176.) In the extreme case of $\varepsilon = 0$ the whole information is lost when the analysis is confined to intra-block, as then $\mathbf{s}'\mathbf{Q}_1 = 0$. It would usually be said in this case that the contrast is "totally confounded" in the intra-block analysis but estimated with "full efficiency" by the inter-block analysis (see Pearce et al., 1974, p.455). In general, however, there is some difficulty in considering $1-\varepsilon$ as the efficiency factor of the design for the contrast $\mathbf{c}'\boldsymbol{\tau} = \mathbf{s}'\mathbf{r}^{\delta}\boldsymbol{\tau}$ estimated in the inter-block analysis, i.e. by (4.10). If there exists a disconnected design which for the vector **s** and the fixed vector **r** satisfies the condition (2.13) of Theorem 2.1 and, simultaneously, the equality $\boldsymbol{\Delta}\boldsymbol{\varphi}_1\boldsymbol{\Delta}'\mathbf{s} = \mathbf{0}$ (i.e. $\boldsymbol{\varphi}_1\boldsymbol{\Delta}'\mathbf{s} = \mathbf{0}$), then, on account of Corollary 3.3(b) and Remark 3.7, it will provide $\mathbf{s}'\mathbf{Q}_2 = \mathbf{s}'\boldsymbol{\Delta}\mathbf{y}$ as the BLUE of $\mathbf{c}'\boldsymbol{\tau}$ under the model (2.1), with the minimum variance of the form (3.33), attainable when the blocks involved by the vector $\mathbf{N}'\mathbf{s}$ are all of equal size. Now, comparing the variance formulae (3.33) and (4.11), it becomes clear that the true inter-block efficiency factor is of the form

$$(1-\varepsilon)\,\frac{k\,\sigma_B^2+(1-K_H^{-1}k)\,\sigma_U^2+\sigma_e^2}{\mathbf{t}'\mathbf{k}^{2\delta}\mathbf{t}\,(\sigma_B^2-K_H^{-1}\sigma_U^2)+\sigma_U^2+\sigma_e^2} \le 1-\varepsilon\,,$$

as $\mathbf{t}'\mathbf{k}^{2\delta}\mathbf{t} \ge k$, provided k is taken equal to $1/\mathbf{t}\mathbf{t}' = \mathbf{s}'\mathbf{N}\mathbf{k}^{-\delta}\mathbf{N}'\mathbf{s}/\mathbf{s}'\mathbf{N}\mathbf{k}^{-2\delta}\mathbf{N}'\mathbf{s}$. So, the coefficient $1-\varepsilon$ gives in general only the upper bound of the efficiency factor of the design for estimating the contrast $\mathbf{c}'\boldsymbol{\tau} = \mathbf{s}'\mathbf{r}^{\delta}\boldsymbol{\tau}$ in the inter-block analysis. It becomes the efficiency factor exactly if and only if $\mathbf{t}'\mathbf{k}^{2\delta}\mathbf{t} = k$, which holds if and only if the involved blocks are all of equal size, i.e. when blocks to which the nonzero elements of the vector **t** correspond are of the same size or, equivalently, all the blocks for which $\sum_{i=1}^{v} s_i n_{ij} \neq 0$ are of the same size, k.

This difficulty with the coefficient $1 - \varepsilon$ considered as an inter-block efficiency factor may be (in addition to the difficulties discussed in Section 3.2) one of the reasons why many authors have been reluctant to use the inter-block analysis to designs that are not of equal block sizes (see, e.g., Pearce, 1983, Section 3.8), and why it is really justifiable to use the term "proper design" for an equiblock- sized design.

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O randomizacyjnej teorii analizy wewnątrzblokowej i międzyblokowej

Streszczenie

W pracy przypomniano ogólny model randomizacyjny dla doświadczeń o układach blokowych i podano warunki otrzymywania najlepszych liniowych estymatorów nieobciążonych w tym modelu. Ponieważ okazuje się, że warunki te są bardzo ograniczające, rozważane jest rozłożenie tego modelu na trzy efektywne podmodele. Znaleziono warunki na istnienie najlepszych liniowych estymatorów nieobciążonych w tych podmodelach. W szczególności pokazano pod jakimi warunkami takie najlepsze estymatory kontrastu parametrów obiektowych otrzymane z analizy wewnątrzblokowej i międzyblokowej estymują bez obciążenia ten sam kontrast. W końcu przedyskutowano zagadnienie współczynników efektywności układu blokowego dla estymacji kontrastu w obu tych analizach.

Słowa kluczowe: najlepsza liniowa estymacja nieobciążona, układy blokowe, współczynnik efektywności, analiza międzyblokowa, analiza wewnątrzblokowa, kwadratowa estymacja nieobciążona o minimalnej normie, model randomizacyjny